PREFACE

General Character and Purpose of the Instructor's Manual

This Manual contains:

(I) Detailed solutions of the even-numbered problems.

(II) General comments on the purpose of each section and its classroom use, with mathematical and didactic information on teaching practice and pedagogical aspects. Some of the comments refer to whole chapters (and are indicated accordingly).

Changes in Problem Sets

The major changes in this edition of the text are listed and explained in the Preface of the book. They include global improvements by updating and streamlining chapters as well as many local improvements aimed at **simplification** of the whole text. Speedy orientation is helped by chapter summaries at the end of the chapters, as in the last edition, and by subdividing sections into subsections with unnumbered headings. Resulting effects of these changes on the problem sets are as follows.

The problems have been changed. The large total number of **over 4000 problems** has been retained, increasing their overall usefulness by the following.

(I) Balancing by extending problem sets that seemed too short and contracting others that were too long, **adjusting the length to the relative importance of the material in a section,** so that important issues are reflected sufficiently well not only in the text but also in the problems. Thus, the danger of overemphasizing minor techniques and ideas is avoided as much as possible.

(II) Simplification by omitting a small number of very difficult problems that appeared in the previous edition, retaining the wide spectrum ranging from simple routine problems to more sophisticated engineering applications, and taking into account the ***"algorithmic thinking"*** that is developing along with computers.

(III) Close amalgamation of text, examples, and problems. This has again been achieved by the **large number of over 600 worked-out examples in the text** and by including problems closely related to those examples.

(IV) Addition of **TEAM PROJECTS, CAS PROJECTS, and WRITING PROJECTS,** whose role is explained in the Preface of the book under *Big Changes.*

These changes in the problem sets will help students in solving problems as well as in gaining a better understanding of practical aspects in the text. It will also enable instructors to explain ideas and methods in terms of examples supplementing and illustrating theoretical discussions—or even replacing some of them if so desired.

"Show the details of your work."

This request repeatedly stated in the book applies to all the problem sets. Of course, it is intended to prevent the student from simply producing answers by a CAS instead of trying to understand the underlying mathematics.

Orientation on Computers

Comments on computer use are included in the Preface of the book. Software systems are listed in the book subsequent to Contents and at the beginning of Chap. 17 on numerical methods.

ERWIN KREYSZIG

Part A. ORDINARY DIFFERENTIAL EQUATIONS

CHAPTER 1 First-Order Differential Equations

Major Changes

Direction fields are now discussed much earlier, in Sec. 1.2. This "geometrical" and "qualitative" approach to differential equations may provide a better conceptual understanding of equations and solutions. The graphical power of a CAS will be helpful in this context.

The second major change concerns the combination of related solution methods. Solution by separation and solution by reduction to separable form now appear in a single section (Sec. 1.3). Similarly, exact equations and integrating factors are both discussed in the same section (Sec. 1.5).

Team Projects and CAS Projects are included in most problem sets.

SECTION 1.1. Basic Concepts and Ideas, page 2

Purpose. To give the student a first impression of what a **differential equation** is and what we mean by **solving** it.

Background Material. For the whole chapter we need integration formulas and techniques, which the student should review.

General Comments

This section should be covered relatively rapidly to get quickly to the actual solution methods in the next sections.

If an example of a partial differential equation is wanted in passing, Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

may be best because of its great physical importance.

Problem Set 1.1 is supposed to help the student with the tasks of

Solving $y' = f(x)$ by calculus,

Finding particular solutions from given general solutions,

Setting up a differential equation for a given function as solution,

Gaining a first experience in modeling, by doing one or two problems,

Gaining a first impression of the importance of differential equations,

without wasting time on matters that can be done much faster, once systematic methods are available.

Comment on "General Solution" and "Singular Solution"

Usage of the term *"general solution"* is not uniform in the literature. Some books use the term to mean a solution that includes *all* solutions, that is, both the particular and the singular ones. We do not adopt this definition for two reasons. First, it is frequently quite

difficult to prove that a formula includes *all* solutions, hence this definition of a general solution is rather useless in practice. Second, *linear* differential equations (satisfying rather general conditions on the coefficients) have no singular solutions (as mentioned in the text), so that for these equations a general solution as defined does include all solutions. For the latter reason, some books use the term "general solution" for linear equations only; but this seems very unfortunate.

Comment on Example 2

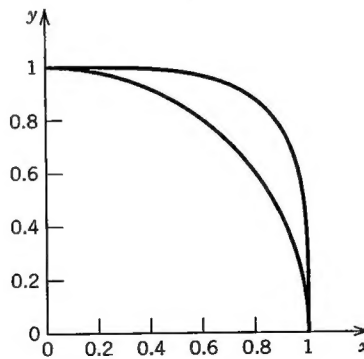
Theoretically inclined students may show (a) by differentiation, (b) directly from the differential equation, that the solution cannot be continued to the closed interval $-1 \leqq x \leqq 1$, where the function is still continuous, but no longer differentiable.

This also illustrates that open intervals generally are the appropriate domains of definition of solutions.

SOLUTIONS TO PROBLEM SET 1.1, page 8

2. $-(\cos 3x)/3 + c$ **4.** $-\frac{1}{2}e^{-x^2} + c$

10. $x - yy' = 0$ by implicit differentiation and division by 2.

12. From the solution and the initial condition, $0 + 1 = c$. *Answer:* $x^4 + y^4 = 1$ $(y > 0)$. The figure shows the portion of this curve in the first quadrant, together with a quarter-circle for comparison.

Section 1.1. Problem 12

14. $y = -x^3/4$ **16.** $y = (\pi/2) \sec x$

18. We get an ellipse with semi-axes $|a|$ and $b = |a|/2$; that is, $x^2/a^2 + y^2/(a/2)^2 = 1$.

20. $e^{-3.6k} = 1/2$, $k = 0.192\,541$, $e^{-k} = 0.825$ after 1 day, $3.012 \cdot 10^{-31}$ after 365 days.

22. $y'' = g$. By two integrations $y' = gt + c_1$ with $c_1 = 0$ because the stone starts from rest, $s = y = gt^2/2 + c_2$ with $c_2 = 0$ because $s(0) = 0$, the stone starts at $s = 0$.

24. k follows from $e^{18000k} = 1/2$, $k = \ln(1/2)/18000 = -0.000038\,508$. *Answer:* $e^{35000k} = 0.26y_0$. Since the decay is exponential, $36000 = 2 \cdot 18000$ would give $(y_0/2)/2 = 0.25y_0$.

26. $y' = ry$, where $r = 0.08$; $y(1)$ equals

1080.00, 1082.43, 1083.28, 1083.29

and $y(5)$ equals

1469.33, 1485.95, 1491.76, 1491.82.

The last two numbers in each line differ only slightly from each other, as claimed.

SECTION 1.2. Geometrical Meaning of $y' = f(x, y)$. Direction Fields, page 10

Purpose. To give the student a feel for the nature of differential equations and the general behavior of fields of solutions. This amounts to a conceptual clarification before entering into formal manipulations of solution methods, the latter being restricted to relatively small—albeit important—classes of equations. This approach is becoming increasingly important, especially because of the graphical power of computer software. It is the analog of conceptual studies of the derivative and integral in calculus as opposed to formal techniques of differentiation and integration.

Comment on Isoclines

These could be omitted since students sometimes confuse them with solutions. In the computer approach to direction fields they no longer play a role.

Comment on Order of Sections

This section could equally well be presented later in Chap. 1, perhaps after one or two formal methods of solution have been studied.

SOLUTIONS TO PROBLEM SET 1.2, page 12

2. $y = x^2 + c$ **4.** $y = ce^{2x}$ **6.** $y = x^3/3 + c$

8. $y = ce^{x} - x - 1$ **10.** $y = x^2/2 + 4$

12. $y = ce^{-x^2/2}$, $c = 1$, a bell-shaped curve

14. The exact solution is $y = 1/(x - 1)$. This is not part of the problem because the solution is obtained by separating variables (which is discussed in the next section), $dy/y^2 = -dx$, $y = 1/(x + c)$, $c = -1$ from the initial condition.

16. $s'(t) = 1/s(t)$. Exact solution $s = +\sqrt{2t + 1}$; this is not part of the problem; it is obtained by separating variables, $s\,ds = dt$, $s^2/2 = t + \tilde{c}$, $s^2 = 2t + c$, and $s(0) = 1$ gives $c = 1$. Now take square roots.

18. The main points of this problem are to realize that a direction field can give general information on solutions and that the present differential equation permits direct conclusions. We can write it

$$\text{(A)} \qquad y' = 4y - y^2 = y(4 - y).$$

We now see that $y = 4$ is a solution. For $x = 0$ Eq. (A) gives

$$y'(0) = y(0)(4 - y(0)).$$

For $0 < y(0) < 4$ both factors on the right are positive, so that $y'(0)$ is positive. Similarly, as long as $0 < y(x) < 4$, we get $y'(x) > 0$, that is, an increasing curve. When $y(x) > 4$, then $y'(x) < 0$ and the curve decreases.

20. CAS PROJECT. (a) The point is that enlargement of subregions may give a more accurate impression.

(b) $y = ce^{-2x}$ is monotone, and the simple direction field may help the student gain confidence in the method. Note that the isoclines are horizontal straight lines $y = const$, a property that should also become visible if the field is produced by computer, without reference to isoclines.

(c) The impression of circles should come out very nicely.

(d) $y' = -x/4y$

SECTION 1.3. Separable Differential Equations, page 14

Purpose. To familiarize the student with the first "big" method by solving simple equations as well as some that require more skill, along with initial value problems (which are simple to solve, once the general solution has been found). Applications of separable equations follow in the next section.

Comment on Example 1

From the implicit solution we can get two explicit solutions

$$y = +2\sqrt{c - (x^2/9)},$$

representing semi-ellipses in the upper half-plane, and

$$y = -2\sqrt{c - (x^2/9)},$$

representing semi-ellipses in the lower half-plane. [Similarly, we can get two explicit solutions $x(y)$ representing semi-ellipses in the left and right half-planes, respectively.] On the x-axis, the tangents to the ellipses are vertical, so that $y'(x)$ does not exist. Similarly for $x'(y)$ on the y-axis.

Comment on Separability

An analytic function $f(x, y)$ in a domain D of the xy-plane can be factored in D, $f(x, y) = g(x)h(y)$, if and only if in D,

$$f_{xy}f = f_x f_y$$

[D. Scott, *American Math. Monthly* **92** (1985), 422–23]. Simple cases are easy to decide, but this may save time in cases of more complicated equations, some of which may perhaps be of practical interest.

SOLUTIONS TO PROBLEM SET 1.3, page 18

2. $25x^2 + y^2 = c$ **4.** $1/(x^3 + c)$ **6.** $1/(kx + c)$

8. $x(u + xu') = xu + x$, $u'x = 1$, $u' = 1/x$, $u = \ln|x| + c = y/x$. *Answer:* $y = x(\ln|x| + c)$.

10. $y + 4x = v$, $y' = v' - 4 = v^2$ by the differential equation. Hence $v' = v^2 + 4$. We may set $v/2 = w$. Then $2w' = 4(w^2 + 1)$. By integration, arc tan $w = 2x + c$, $w = \tan(2x + c) = v/2 = y/2 + 2x$. *Answer:* $y = -4x + 2\tan(2x + c)$.

12. $yy' = -x$, $y^2/2 = -x^2/2 + \tilde{c}$, $x^2 + y^2 = c$. From this and the initial condition, $1^2 + (\sqrt{3})^2 = 4 = c$. *Answer:* $x^2 + y^2 = 4$, a circle of radius 2.

14. By integration, $y^4/4 + x^4/4 = c$. From this and the initial condition, $1/4 + 0 = c$. *Answer:* $x^4 + y^4 = 1$.

16. By separation of variables, $dy/(1 + 4y^2) = dx$. We may set $2y = z$, hence $y' = z'/2$. By substitution,

$$\frac{dz}{2(1 + z^2)} = dx, \qquad \text{arc tan } z = 2x + c, \qquad z = \tan(2x + c),$$

hence $y = z/2 = \frac{1}{2}\tan(2x + c)$. From this and the initial condition, $0 = \frac{1}{2}\tan c$, $c = 0$. *Answer:* $y = \frac{1}{2}\tan 2x$.

18. By separation, integration, and exponentiation,

$$\frac{dr}{r} = -2t\,dt, \qquad \ln r = -t^2 + \tilde{c}, \qquad r = ce^{-t^2}.$$

From this and the initial condition, $r(0) = c = 2.5$. *Answer:* $r = 2.5e^{-t^2}$.

20. Substitute $y/x = u$, $y = xu$, $y' = u + xu'$ and simplify to get

$$x(u + xu') = x^3(u-1)^3 + xu, \qquad u' = x(u-1)^3.$$

By separation and integration,

$$\frac{du}{(u-1)^3} = x\,dx, \qquad -\frac{1}{2(u-1)^2} = \frac{x^2}{2} + \tilde{c}, \qquad (u-1)^2 = \frac{1}{c - x^2}.$$

Hence by taking roots, $y = xu = x + x/\sqrt{c - x^2}$. From this and the initial condition, $3/2 = 1 + 1/\sqrt{c-1}$, $c = 5$. *Answer:*

$$y = x + \frac{x}{\sqrt{5 - x^2}}.$$

22. We substitute $y/x = u$, $y = xu$, $y' = u + xu'$ and simplify, obtaining

$$x(u + xu') = xu + x^2 \sec u, \qquad u' = \sec u, \qquad \cos u\,du = dx, \qquad \sin u = x + c.$$

From this and the initial condition $y(1) = \pi$ we have $u(1) = \pi$, $0 = \sin \pi = 1 + c$, $c = -1$. *Answer:* $y = x \arcsin(x - 1)$.

24. $v = x + y - 2$, $y' = v' - 1 = v^2$, $v' = v^2 + 1$ can be separated, $\dfrac{dv}{v^2 + 1} = dx$, $\arctan v = x + c$, $y = v - x + 2 = 2 - x + \tan(x + c)$.

26. TEAM PROJECT. **(a)** Note that at the origin, $x/y = 0/0$, so that y' is undefined at the origin.

(b) $(xy)' = y + xy' = 0$, $y' = -y/x$

(c) $y = cx$. Here the student should learn that c must not appear in the differential equation. $y/x = c$, $y'/x - y/x^2 = 0$, $y' = y/x$.

(d) The right sides $-x/y$ and y/x are the slopes y' of the curves. Orthogonality is important and will be discussed further in Sec. 1.8.

SECTION 1.4. Modeling: Separable Equations, page 19

Purpose. This section contains some typical applications to choose from, depending on students' interests and background. They serve to convince the student of the practical importance of differential equations. Similarly, Problem Set 1.4 contains much more material than one would ordinarily wish to discuss.

Comment on Example 4

Although Newton's second law involves acceleration, hence a second derivative, it is often possible to stay within first-order equations, as in this case.

Comment on Footnote 4

Newton conceived his method of fluxions (calculus) in 1665–1666, at the age of 22. *Philosophiae Naturalis Principia Mathematica* was his most influential work.

Leibniz invented calculus independently in 1675 and introduced notations that were essential to the rapid development in this field. His first publication on differential calculus appeared in 1684.

SOLUTIONS TO PROBLEM SET 1.4, page 23

2. $y'' = k$ (constant acceleration). By two integrations, $y = \frac{1}{2}kt^2 + 10t$, where we used the given initial speed. After 50 sec we have $y(50) = 1250k + 500 = 2000$. This gives $k = 1.2$. By differentiation and substitution, $y'(50) = 50k + 10 = 70$ meters/sec $= 252$ km/hour.

4. Acceleration $y'' = 7t$. Hence $y' = 7t^2/2$, $y = 7t^3/6$, $y'(10) = 350$ (initial speed of further flight = end speed upon return from peak), $y(10) = 7000/6 = 1167$ (height reached after the 10 sec). At the peak, $v = 0$, $s = 0$, say; thus for the further flight (measured from the peak), $s(t) = (g/2)t^2 = 4.9t^2$, $v(t) = 9.8t = 350$ (see before). This gives the further flight time to the peak $t = t_1 = 350/9.8 = 35.7$ and the further height $s(t_1) = 4.9t_1^2 = 6245$. *Answer:* $1167 + 6245 = 7412$ [m].

6. $e^{-k\cdot 10} = \frac{1}{2}$, $k = \frac{1}{10}\ln\frac{1}{2} = 0.069315$, $e^{-kt_0} = 0.01$ (1% is the remaining moisture). *Answer:* $t_0 = \dfrac{1}{k}\ln 100 = 66.4$ min; practically 1 hour.

8. The acceleration is $a = 9 \cdot 10^6$ meters/sec^2, and the distance traveled is 5.5 meters. This is obtained as follows. Since $s(0) = 0$ (i.e., we count time from the instant the particle enters the accelerator), we have for a motion of constant acceleration

$$\text{(A)} \qquad s(t) = a\frac{t^2}{2} + bt$$

and the velocity is

$$v(t) = s'(t) = at + b.$$

From the given data we thus obtain $v(0) = b = 10^3$ and

$$v(10^{-3}) = 10^{-3}a + 10^3 = 10^4$$

so that

$$a = 10^3(10^4 - 10^3) = 10^7 - 10^6 = 9 \cdot 10^6.$$

Finally, with this a and that b, from (A) we get

$$s(10^{-3}) = 9 \cdot 10^6 \cdot \frac{10^{-6}}{2} + 10^3 \cdot 10^{-3} = 5.5 \text{ [m]}.$$

10. At the earth's surface the minimum velocity for escape is v_0^2, and from (11) and (12) in Example 4 we see that then the square of the velocity at any distance r from the center of the earth is

$$v(r)^2 = \frac{2gR^2}{r} + v_0^2 - 2gR = \frac{2gR^2}{r}$$

so that at the point of separation $r_0 = R + 1000$ (i.e., 1000 km above the earth's surface) the projectile has the velocity

$$v(r_0) = \sqrt{\frac{2gR^2}{r_0}} = \sqrt{\frac{2 \cdot 0.0098 \cdot 6372^2}{7372}} = \sqrt{107.95} = 10.39 \text{ [km/sec]}.$$

This is the minimum velocity of escape at the separation point.

12. $\Delta A = -kA\Delta x$ (A = amount of incident light, ΔA = absorbed light, Δx = thickness, $-k$ = constant of proportionality). Let $\Delta x \to 0$. Then $A' = -kA$. Hence $A(x) = A_0e^{-kx}$ is the amount of light in a thick layer at depth x from the surface of incidence.

14. Let $y(t)$ be the amount of salt in the tank at time t. Then each gallon contains $y/400$

lb of salt. $2\Delta t$ gal of water run in during a short time Δt, and $-\Delta y = 2\Delta t(y/400) = \Delta t\, y/200$ is the loss of salt during Δt. Thus $\Delta y/\Delta t = -y/200$, $y' = -0.005y$, $y(t) = 100e^{-0.005t}$. *Answer:* $y(60) = 100e^{-0.3} = 74$ [lb].

16. Let $V = V(t)$ be the volume and $r = r(t)$ the radius. Then the area is $A = 4\pi r^2$. The rate of change dV/dt is proportional to A; thus by the chain rule, denoting the constant of proportionality by k,

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt} = kA = 4k\pi r^2.$$

At $t = 0$ the radius is 1 and after 2 months, it is $\frac{1}{2}$. Now dividing the previous equation by $4\pi r^2$ and integrating, we obtain

$$\frac{dr}{dt} = k, \qquad r = kt + c$$

and that condition can be used to find k and c,

$$r(0) = c = 1, \qquad r = kt + 1, \qquad r(2) = 2k + 1 = \tfrac{1}{2}.$$

Hence $k = -1/4$, and from this and the condition that the ball have radius 0.05 cm, we obtain

$$0.05 = r(t) = -\tfrac{1}{4}t + 1, \qquad \text{thus} \qquad t = 4 \cdot 0.95 = 3.8.$$

The *answer* is 3.8 months.

18. $W = mg$ in Fig. 12 is the weight (the force of attraction acting on the body). Its component parallel to the surface is $mg \sin \alpha$, and $N = mg \cos \alpha$. Hence the friction is $0.2mg \cos \alpha$, and it acts against the direction of motion. From this and Newton's second law, noting that the acceleration is dv/dt (v the velocity), we obtain

$$\begin{aligned} m\frac{dv}{dt} &= mg \sin \alpha - 0.2mg \cos \alpha \\ &= m \cdot 9.80(0.500 - 0.2 \cdot 0.866) \\ &= 3.203m. \end{aligned}$$

The mass m drops out, and two integrations give

$$v = 3.203t \qquad \text{and} \qquad s = 3.203\frac{t^2}{2}.$$

Since the slide is 10 meters long, the last equation with $s = 10$ gives the time

$$t = \sqrt{2 \cdot 10/3.203} = 2.50.$$

From this we obtain the *answer*

$$v = 3.203 \cdot 2.50 = 8.01 \text{ [meters/sec]}.$$

20. TEAM PROJECT. (a) This property is worth noting. It is obtained by substituting $t = t(h)$ into $v(t)$; thus

$$a = g, \qquad v = gt, \qquad h = gt^2/2, \qquad t = \sqrt{2h/g}, \qquad v = g\sqrt{2h/g} = \sqrt{2gh}.$$

(b) This is a typical exercise in modeling. It is remarkable that A and $B(h)$ in (14) are unspecified, so that (14) could serve as a model for various types of tanks (cylindrical, conical, hemispherical, etc.). $B\Delta h$ is the decrease in volume when h

decreases by Δh during a short time Δt, and this must equal $Av\Delta t$, which, by Torricelli's law equals $A \cdot 0.600\sqrt{2gh}\Delta t$. By equating the two expressions, and introducing a minus sign (since the water level decreases), we get

$$B\Delta h = -26.56A\sqrt{h}\Delta t,$$

Dividing by Δt and letting $\Delta t \to 0$ gives (14).

(c) This is the simplest case because B is constant (independent of h), and we can easily solve (14) by separation of variables and integration,

$$h^{-1/2}\,dh = -26.56(A/B)\,dt$$
$$h^{1/2} = -13.28(A/B)t + c.$$

(d) $A/B = (1/100)^2$, $\sqrt{h(0)} = \sqrt{150} = 12.25 = c$, and the tank will be empty at t satisfying

$$0 = -0.001328t + 12.25; \qquad \text{that is,} \qquad t = 9924 \text{ sec} = 154 \text{ min}.$$

SECTION 1.5. Exact Differential Equations. Integrating Factors, page 25

Purpose. This is the second "big" method in this chapter, after separation of variables, and also applies to equations that are not separable. The criterion (5) is basic. Simpler cases are solved by inspection, more involved cases by integration, as explained in the text.

Comment on Footnote 12

Condition (5) is equivalent to (6″) in Sec. 9.2, which is equivalent to (6) in the case of two variables x, y. Simple connectedness of D follows from our assumptions in Sec. 1.5. Hence the differential form is exact by Theorem 3, Sec. 9.2, part (b) and part (a), in that order.

Method of Integrating Factors

This greatly increases the usefulness of solving exact equations. It is important in itself as well as in connection with linear equations in the next section. Problem Set 1.5 will help the student gain skill needed in finding integrating factors. Inasmuch, the method has somewhat the flavor of tricks, but on the other hand, Theorems 1 and 2 show that at least in some cases one can proceed systematically—and one of them is precisely the case needed in the next section.

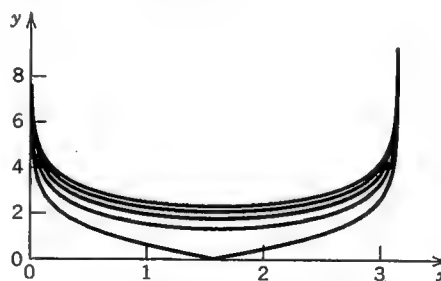
Comment on Notation in Example 5

The standard notation for $(\sin y)^2$ is $\sin^2 y$, hence $\sin y^2$ clearly means $\sin(y^2)$; the parentheses are superfluous, but we wanted to help poorer students.

SOLUTIONS TO PROBLEM SET 1.5, page 31

2. $2x\,dx - 2y\,dy = 0$, hyperbolas, with asymptotes $y = \pm x$

4. $-(2x\,dx + 2y\,dy)/(x^2 + y^2)^2 = 0$, concentric circles

6. $\cos x \cosh y\,dx + \sin x \sinh y\,dy = 0$. The curves $u = const$ go vertically upward to infinity as $\sin x \to 0$, as $x \to 0, \pm\pi, \cdots$; see the figure.

Section 1.5. Problem 6

8. $y/x = c$

10. $re^{3\theta} = c$

12. $x \cot y + x^3/3 = c$

14. Yes, $y = 3.8 \sin 2x$

16. Yes, $y^2 + ye^x = 0$

18. $(\omega \cos \omega y)_x = 0$ but $(2 \sin \omega y)_y \neq 0$. Not exact. By separation of variables,

$$\cot \omega y \, dy = -\frac{2}{\omega} dx, \qquad \frac{1}{\omega} \ln |\sin \omega y| = -\frac{2}{\omega} x + \tilde{c}, \qquad \sin \omega y = ce^{-2x}.$$

Now $y(0) = \pi/(2\omega)$ gives $\sin(\pi/2) = c$. Hence $c = 1$. *Answer:* $e^{2x} \sin \omega y = 1$.

20. $(2xye^{x^2})_y = 2xe^{x^2} = (e^{x^2})_x$ shows exactness. By integration,

$$ye^{x^2} = c.$$

$y(0) = 2$ gives $c = 2$. *Answer:* $y = 2e^{-x^2}$.

22. Equation (9) becomes $s^4 + t^4 = const$; see the figure in the solution to Prob. 12 of Sec. 1.1 in this Manual.

24. $(xy)^{-1} \, dy - x^2 \, dx = 0$ has the integrating factor $F = y$, giving

$$\text{(A)} \qquad x^{-1} \, dy - x^{-2} y \, dx = 0,$$

which is exact because

$$(x^{-1})_x = -x^{-2} = (-x^{-2}y)_y.$$

Now (A) implies

$$x^{-1} \, dy - x^{-2} y \, dx = \frac{x \, dy - y \, dx}{x^2} = d\left(\frac{y}{x}\right) = 0$$

so that

$$\frac{y}{x} = c, \qquad y = cx$$

as claimed, but $y \equiv 0$ is not a solution of the original equation.

26. $y \cos(x + y) \, dx + [y \cos(x + y) + \sin(x + y)] \, dy = 0$ is exact because

$$\begin{aligned} [y \cos(x + y)]_y &= \cos(x + y) - y \sin(x + y) \\ &= [y \cos(x + y) + \sin(x + y)]_x. \end{aligned}$$

By inspection or systematically,

$$y \sin(x + y) = c.$$

28. The new equation is

$$3(y+1)^2x^{-4}\,dx - 2(y+1)x^{-3}\,dy = 0.$$

It is exact,

$$M_y = N_x = 6(y+1)x^{-4}.$$

The general solution is

$$(y+1)^2x^{-3} = c.$$

30. The new equation is

$$2\cos 2x \cos y\,dx - \sin 2x \sin y\,dy = 0.$$

It is exact,

$$M_y = N_x = -2\cos 2x \sin y.$$

A general solution is

$$\sin 2x \cos y = c.$$

32. $F = y^2$ gives the new equation

$$2xy^3\,dx + 3x^2y^2\,dy = d(x^2y^3) = 0,$$

which is exact and has the general solution $x^2y^3 = const.$

34. $F = e^{2x}$ gives the new equation

$$e^{2x}(2\cos y\,dx - \sin y\,dy) = 0.$$

This equation is exact,

$$M_y = N_x = -2e^{2x}\sin y.$$

The general solution is $e^{2x}\cos y = c.$

36. $F = 1/(x+1)(y+1)$ gives the exact equation

$$\frac{dx}{x+1} - \frac{dy}{y+1} = 0.$$

The general solution is

$$y+1 = c(x+1).$$

38. WRITING PROJECT. Suitable equations abound; for instance, the equations

$$xy' + y + 4 = 0, \qquad b^2x\,dx + a^2y\,dy = 0,$$

etc. can be solved by inspection, separation, or as exact equations.

40. CAS PROJECT. (a) Theorem 1 does not apply. Theorem 2 gives

$$\frac{1}{F}\frac{dF}{dy} = \frac{-1}{y^2 \sin x}(0 + 2y\sin x) = -\frac{2}{y}, \qquad F = \exp\int -\frac{2}{y}\,dy = \frac{1}{y^2}.$$

The exact equation is

$$y^{-2}\,dy - \sin x\,dx = 0,$$

as one could have seen by inspection—any equation of the form

$$f(x)\,dx + g(y)\,dy = 0$$

is exact! We now obtain

$$u = \int -\sin x\,dx = \cos x + k(y)$$

$$u_y = k'(y) = \frac{1}{y^2}, \qquad k = -\frac{1}{y},$$

$$u = \cos x - \frac{1}{y} = c.$$

(b) Yes,

$$y' = y^2 \sin x, \qquad \frac{dy}{y^2} = \sin x\, dx, \qquad -\frac{1}{y} = -\cos x + c, \qquad y = \frac{1}{\cos x + \tilde{c}}.$$

(c) The vertical asymptotes that some CAS programs draw disturb the graph. From the solution in (b) the student should conclude that for each initial condition $y(x_0) = y_0$ with $y_0 \neq 0$ there is a unique particular solution because from (b),

$$\tilde{c} = \frac{1 - y_0 \cos x_0}{y_0}.$$

(d) $y \equiv 0$

SECTION 1.6. Linear Differential Equations. Bernoulli Equation, page 33

Purpose. Linear equations are of great practical importance, as Problem Set 1.6 illustrates (and even more so are higher order linear equations in Chap. 2). We show that the homogeneous equation is easily separated and the nonhomogeneous equation is solved, once and for all, in the form of an integral (4) by the method of integrating factors. Of course, in simpler cases one does not need (4), as our examples illustrate.

Comment on Notation

We write

$$y' + p(x)y = r(x).$$

$p(x)$ seems standard. $r(x)$ suggests *"right side."* The notation

$$y' + p(x)y = q(x)$$

used in some calculus books (which are not concerned with higher order equations) would be short-sighted here because a few weeks later in Chap. 2 we turn to second-order equations

$$y'' + p(x)y' + q(x)y = r(x),$$

where we need $q(x)$ on the left, thus in a quite different role (and on the right we would have to choose another letter different from that used in the first-order case).

Comment on Content

Bernoulli's equation appears occasionally in practice, so the student should remember how to handle it.

Riccati and **Clairaut equations** are less important than Bernoulli's, so we have put them in the problem set; they will not be needed in our further work.

Input and **output** have become common terms in various contexts, so we thought it a good place to mention them here.

Problems 23–30 express the properties that make **linearity** important, and their counterparts will, of course, reappear in Chap. 2.

Comment on Footnote 14

Eight members of the Bernoulli family became known as mathematicians; for more details, see p. 220 in Ref. [2] listed in Appendix 1.

SOLUTIONS TO PROBLEM SET 1.6, page 38

4. $y = ce^{-2x} + 1.25$

6. $y = ce^{-x^2/2} + 4$. Separation of variables seems simplest here, $y' = -(y - 4)x$; then divide by $y - 4$, etc.

8. $y = ce^{-4x} + \frac{4}{17}\cos x + \frac{1}{17}\sin x$. The particular solution can be obtained by substituting $y = a\cos x + b\sin x$, which leads to $a = 4b$ and $17b = 1$ by comparing with $\cos x$ on the right of the given equation. This avoids the integration.

10. $y = e^{-x}(c - \ln|\cos x|)$

12. $y = x^{-3}(\ln|x| + c)$

14. $x^2y' + 2xy = (x^2y)' = \sinh 5x$; now integrate to get

$$x^2y = \tfrac{1}{5}\cosh 5x + c, \qquad \text{thus} \qquad y = x^{-2}(\tfrac{1}{5}\cosh 5x + c).$$

These problems illustrate that the integral solution formula (4) can be avoided in many cases.

16. $y = cx^3e^x - x$ is the general solution. The initial condition gives $c = 1$.

18. This homogeneous linear equation has the general solution $y = c\sec x$, and $c = -2$ from the initial condition.

20. $y = e^{-2x^3}(c - 1/x)$ is the general solution; $c = 1$ from the initial condition.

22. $y = cx^{-4} + x^4$ is the general solution. The initial condition gives $c = 1$.

24. Problems 23–30 require proofs by substitution, so they are basically very similar. By working these problems the student should become aware of the difference between homogeneous and nonhomogeneous equations. This will also serve as a preparation for the corresponding theorems for higher order equations, some of which are important in constructing general solutions of nonhomogeneous equations from those of homogeneous equations.

30. $y' + p_0y = r_0$, $y' = -p_0(y - r_0/p_0)$, $y'/(y - r_0/p_0) = -p_0$, $\ln(y - r_0/p_0) = -p_0x + \tilde{c}$, $y = r_0/p_0 + ce^{-p_0x}$

32. $u = y^2$, $yy' + y^2 = -x$, $\frac{1}{2}u' + u = -x$, $u' + 2u = -2x$; hence

$$u = e^{-2x}\left[-\int e^{2x}\, 2x\, dx + c\right] = \tfrac{1}{2} - x + ce^{-2x}, \qquad y = \sqrt{u}$$

34. This differential equation can simply be solved by separating variables,

$$\cot y\, dy = \frac{dx}{x - 1}, \qquad \ln|\sin y| = \ln|x - 1| + \tilde{c},$$
$$y = \arcsin[c(x - 1)] \qquad \text{or} \qquad x = 1 + \hat{c}\sin y.$$

As an alternative, we can regard it as a differential equation for the unknown function $x = x(y)$ and solve it by formula (4) with x and y interchanged.

36. Take x as the dependent variable to get

$$\frac{dx}{dy} = y^{-2}(\sinh 3y - 2xy), \qquad \frac{dx}{dy} + \frac{2x}{y} = \frac{1}{y^2}\sinh 3y$$

and by (4) with x and y interchanged,

$$x = y^{-2}\left[\int y^2(\sinh 3y)y^{-2}\,dy + c\right]$$
$$= y^{-2}[\tfrac{1}{3}\cosh 3y + c].$$

38. Using the given transformation $y^2 = z$, we obtain the linear differential equation

$$z' + \left(1 - \frac{1}{x}\right)z = xe^x,$$

which we can solve by (4) with z instead of y,

$$z = xe^{-x}\left(\int \frac{1}{x}\, e^x xe^x\,dx + c\right) = xe^{-x}(\tfrac{1}{2}e^{2x} + c) = cxe^{-x} + \tfrac{1}{2}xe^x.$$

From this we obtain $y = \sqrt{z}$.

40. $y' + y = 1 - \cos(\pi t/12)$. *Solution:*

$$y = e^{-t}\left[\int e^t\left(1 - \cos\frac{\pi t}{12}\right)dt + c\right]$$
$$= e^{-t}\left[e^t\left(1 - 0.936\cos\frac{\pi t}{12} - 0.245\sin\frac{\pi t}{12}\right) + c\right]$$

where $y(0) = 2$ gives $c = 1.936$, so that we get the *answer*

$$y = 1.936e^{-t} + 1 - 0.936\cos\frac{\pi t}{12} - 0.245\sin\frac{\pi t}{12}.$$

42. From (4) and the initial condition $v(0) = 0$ we obtain

$$v(t) = \frac{W - B}{k}(1 - e^{-kt/m}), \qquad \text{where} \qquad m = \frac{W}{g} = \frac{2254}{9.80} = 230 \text{ [kg]}.$$

By integration, using $y(0) = 0$,

$$y(t) = \frac{W - B}{k}\left[t - \frac{m}{k}(1 - e^{-kt/m})\right].$$

From $v(t)$ we calculate that $v = v_{\text{crit}}$ when

$$t = t_{\text{crit}} = \frac{m}{k}\ln\frac{1}{1 - 12k/(W - B)} = 17.2 \text{ [sec]}.$$

This gives $y(t_{\text{crit}}) = 105$ meters, approximately.

44. The given equation $y' = x^3(y - x)^2 + x^{-1}y$ shows immediately that $y = x$ is a solution. It is a Riccati equation; its standard form is

$$y' + (2x^4 - x^{-1})y = x^3y^2 + x^5,$$

as follows by direct calculation.

From $w = y - x$ we have $y = w + x$, and from the given equation we get

$$y' = w' + 1 = x^3w^2 + x^{-1}(w + x), \qquad w' - x^{-1}w = x^3w^2,$$

a Bernoulli equation. To solve it, set $1/w = z$, $w = 1/z$, $w' = -z'/z^2$ and by substitution

$$-\frac{z'}{z^2} - \frac{1}{zx} = \frac{x^3}{z^2}, \qquad z' + \frac{1}{x}z = -x^3.$$

This linear equation can now be solved by (4),

$$z = \frac{1}{x}\left[\int x(-x^3)\,dx + c\right] = \frac{1}{x}\left[-\frac{1}{5}x^5 + c\right].$$

Answer:

$$y = w + x = \frac{1}{z} + x = \left(cx^{-1} - \frac{1}{5}x^4\right)^{-1} + x.$$

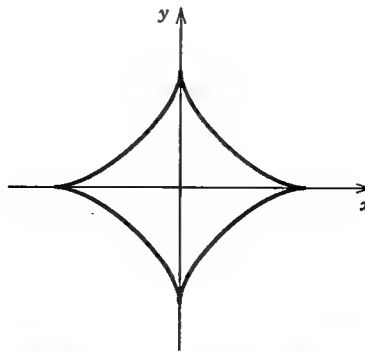
46. By differentiation, $y' = y' + xy'' - y''/y'^2 = y' + y''(x - 1/y'^2)$. Now (A) $y'' = 0$ gives $y = cx + a$ and $a = 1/c$ by substitution, a family of straight lines. (B) $x = 1/y'^2$ gives by integration $y = 2x^{1/2} + \tilde{c}$ and $\tilde{c} = 0$ by substituting y and y' into the given equation, hence $y = 2\sqrt{x}$, the singular solution, to which the straight lines in (A) are tangent.

48. $y = ax + b$ intersects the axes at $(-b/a, 0)$ and $(0, b)$. Length 1 implies that $b = -a/\sqrt{1 + a^2}$. Now $a = y'$ and we get the equation given in the problem. We write $y' = s$. Then the singular solution results from $x = -g'(s) = (1 + s^2)^{-3/2}$. From this and the differential equation, by simplification,

$$y = sx - s/(1 + s^2)^{1/2} = -s^3/(1 + s^2)^{3/2}.$$

The result is

$$x = (1 + s^2)^{-3/2}, \qquad y = -s^3(1 + s^2)^{-3/2},$$

a parametric representation of the astroid (see the figure). Adding these two expressions, each raised to the power 2/3, we get the formula in the problem.

Section 1.6. Astroid in Problem 48

SECTION 1.7. Modeling: Electric Circuits, page 41

Purpose. To model by Kirchhoff's laws those circuits (*RL* and *RC*) that lead to a first-order equation and to discuss the currents for the simplest inputs (constant and sinusoidal). This is a major standard application of linear equations and will help *all* students—not just electrical engineers—to gain further experience in modeling. (*RLC*-circuits, leading to second-order equations, follow in Sec. 2.12.)

Shorter Courses. Sections 1.7–1.9 may be omitted without interrupting continuity.

SOLUTIONS TO PROBLEM SET 1.7, page 47

2. δ increases with L, indicating that L has a retarding effect (an inertia effect—to be studied further in Sec. 2.12, where we show the analogy between mechanical and electrical quantities).

4. Solve (4) algebraically for $I' = dI/dt$; this gives $I' = [E_0 - RI]/L$, and $I' > 0$ as long as $E_0/R > I(t)$, so that $I(t)$ begins to increase when $I(0) < E_0/R$. Similarly, $I(0) > E_0/R$ implies that $I(t)$ begins to decrease.

6. For $t = 1$ and $I = 0.99E_0/R$ we have from (5**) with $L = 10$

$$0.99\,\frac{E_0}{R} = \frac{E_0}{R}(1 - e^{-R/10}), \quad e^{-R/10} = 0.01,$$

$$e^{R/10} = 100, \quad R = 10 \ln 100 = 46 \text{ [ohms]}.$$

8. We obtain

$$RA + \omega LB = 0, \qquad -\omega LA + RB = E_0$$

and from this,

$$A = -\omega LE_0/(R^2 + \omega^2L^2), \qquad B = RE_0/(R^2 + \omega^2L^2).$$

By (14), Appendix A3.1,

$$\sqrt{A^2 + B^2} = \sqrt{\frac{\omega^2L^2E_0{}^2 + R^2E_0{}^2}{(R^2 + \omega^2L^2)^2}} = \frac{E_0}{\sqrt{R^2 + \omega^2L^2}},$$

as in (6), and $\tan\delta = -A/B = \omega L/R$.

10. TEAM PROJECT. **(a)** $I(t)$ is continuous. A jump J/L of I' gives a jump J of LI', equal to the jump of $E(t)$ on the right side of (4).

(b) $I = I_1 = 1 - \frac{1}{2}e^{-t}$ when $0 \leqq t \leqq 4$, $I_2 = c_2e^{-(t-4)}$ (this is better than writing c_2e^{-t}, which, however, would also work). Now by continuity of I, $I_2(4) = I_1(4) = 1 - \frac{1}{2}e^{-4} = 0.99$.

(c) Here we proceed as in (b), with letters instead of numbers.

$$I_1(t) = \frac{E_0}{R} + c_1e^{-Rt/L} \quad \text{when} \quad 0 \leqq t \leqq a,$$

$$I_1(0) = \frac{E_0}{R} + c_1 = I_0, \qquad c_1 = I_0 - \frac{E_0}{R}, \qquad I_2(t) = c_2e^{-R(t-a)/L}.$$

Again, $t - a$ in the exponent is practical, but we could use t instead. c_2 follows from

$$I_2(a) = c_2 = I_1(a) = \frac{E_0}{R} + \left(I_0 - \frac{E_0}{R}\right)e^{-Ra/L}.$$

12. From the first line in (11),

$$I(0) = c + \frac{\omega E_0 C}{1 + (\omega RC)^2} = 0;$$

this gives c. Inserting it into the first line, we have

$$I(t) = \frac{\omega E_0 C}{1 + (\omega RC)^2}\left[-e^{-t/RC} + \cos\omega t + \omega RC \sin\omega t\right].$$

14. Differentiation of (7) gives

$$RI' + R'I + \frac{I}{C} = 0, \qquad (200 - t)I' + 3I = 0, \qquad \frac{dI}{I} = \frac{3}{t - 200}\,dt.$$

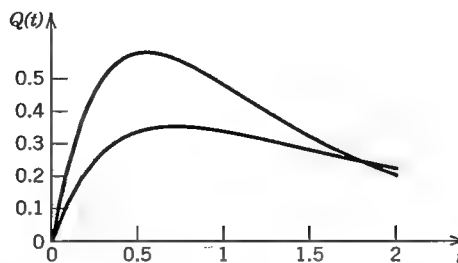
Hence $I = c(t - 200)^3$ and $I(0) = 1$ gives $c = -1/(8 \cdot 10^6)$, thus

$$I(t) = (200 - t)^3/(8 \cdot 10^6) \quad \text{if } 0 \leqq t \leqq 200, \qquad I(t) = 0 \text{ if } t > 200.$$

16. We want the time such that $Q = Q_0 e^{-t/RC} = 0.01Q_0$. Hence $t = RC \ln 100 = 4.605RC$ [sec].

18. $20Q' + 10Q = 30e^{-3t}$ is the new equation. For the initial condition $Q(0) = 0$ we obtain the particular solution

$$Q(t) = 0.6(e^{-t/2} - e^{-3t}).$$

$Q'(t) = 0.6(-0.5e^{-t/2} + 3e^{-3t}) = 0$ gives $e^{2.5t} = 6$, $t_m = (\ln 6)/2.5 = 0.717$ [sec] and $Q_m = 0.349$ [coulomb]. The larger R has caused a smaller Q_m at a later time t_m. See the figure, where the upper curve corresponds to Prob. 17.

Section 1.7. Problem 18

20. TEAM PROJECT. **(a)** Use (3*), (3). Then from (7),

$$RI(t) = E(t) - Q(t)/C.$$

Divide by R and set $t = 0$.

(b) This follows from (7). The charge on the capacitor cannot change abruptly. Hence RI on the left must have a jump of magnitude J, so that I must have a jump J/R.

(c) $I(0) = 0$ by (a). $I' + I = t$, $I = t - 1 + e^{-t}$, $I(2) = 1 + e^{-2}$. dE/dt has a jump -2 at $t = 2$. Hence the current I_2 for $t \geqq 2$ satisfies

$$I_2' + I_2 = 0, \qquad I_2(2) = -1 + e^{-2}.$$

Solution: $I_2 = (1 - e^2)e^{-t}$.

SECTION 1.8. Orthogonal Trajectories of Curves. *Optional,* page 48

Purpose. To show that families of curves $F(x, y, c) = 0$ can be described by differential equations $y' = f(x, y)$ and the switch to $y' = -1/f(x, y)$ produces as general solution the orthogonal trajectories. This is a nice application, which may also help the student gain more self-confidence, skill, and a deeper understanding of the nature of differential equations. We leave this section ***optional,*** for reasons of time. This will cause no gap.

SOLUTIONS TO PROBLEM SET 1.8, page 51

2. $(x - c)^2 + (y - c^3)^2 - 4 = 0$ gives a circle of radius 2 with center $(x_0, y_0) = (c, c^3)$, and we see that the coordinates of the center satisfy $y_0 = x_0{}^3$, as required.

4. $y' = 1/2$. This is the slope of these parallel straight lines.

6. From the given formula we get arc tan $y = x + c$. Differentiating and applying the chain rule, we obtain

$$\frac{y'}{1 + y^2} = 1. \qquad \textit{Answer:} \qquad y' = 1 + y^2.$$

8. $x^{-4}y = c$. By differentiation, $-4x^{-5}y + x^{-4}y' = 0$. Algebraic solution for y' gives the *answer*

$$y' = 4y/x.$$

10. From the given representation we get

$$ye^{-x^2} = c, \qquad e^{-x^2}(y' - 2xy) = 0, \qquad y' = 2xy.$$

This is the differential equation of the given family of curves. From this we have the differential equation of the orthogonal trajectories

$$y' = -\frac{1}{2xy}.$$

Separation of variables and integration gives

$$2y\,dy = -\frac{dx}{x}, \qquad y^2 = -\ln|x| + \tilde{c} = \ln\frac{1}{|x|} + \tilde{c}.$$

Taking exponentials and solving for x as a function of y, we obtain

$$e^{y^2} = \frac{c^*}{x}, \qquad x = c^* e^{-y^2}.$$

These are ***bell-shaped curves***—note that in Sec. 1.3 the roles of x and y are interchanged.

12. $y' = 1/x$ gives for the orthogonal trajectories $y' = -x$ with the solution

$$y = -\tfrac{1}{2}x^2 + c^*.$$

Note that here we have congruent curves as well as congruent orthogonal trajectories.

14. Squaring the given formula, differentiating, and solving algebraically for y', we obtain

$$y^2 - x = c, \qquad 2yy' = 1, \qquad y' = \frac{1}{2y}.$$

This is the differential equation of the given curves. Hence the differential equation of the orthogonal trajectories is

$$y' = -2y.$$

By separation of variables and integration we obtain

$$\ln|y| = -2x + \tilde{c}.$$

Exponentiation gives the *answer*

$$y = c^* e^{-2x}.$$

16. Differentiating the given formula, we obtain

$$xy' + y = 0. \qquad \text{Thus} \qquad y' = -\frac{y}{x}.$$

This is the differential equation of the given hyperbolas. Hence the differential equation of the orthogonal trajectories is

$$y' = \frac{x}{y}.$$

Separation of variables and integration gives

$$y\,dy = x\,dx, \qquad \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + \tilde{c}.$$

Answer: The hyperbolas $x^2 - y^2 = c^*$ are the orthogonal trajectories of the given hyperbolas.

18. $x + x^{-1}y^2 = 2c$ by algebra. By differentiation,

$$1 - x^{-2}y^2 + 2x^{-1}yy' = 0, \qquad \text{thus} \qquad y' = \frac{y^2 - x^2}{2xy}.$$

Hence the equation of the trajectories is

$$\frac{dx}{dy} = \frac{x^2 - y^2}{2xy}.$$

To solve it for $x = x(y)$, set $v = x/y$ and separate.

$$\frac{2v}{v^2 + 1}\,dv = -\frac{dy}{y}. \qquad \text{Then} \qquad \ln(v^2 + 1) = -\ln|y| + \tilde{c}$$

which gives

$$v^2 + 1 = \frac{x^2}{y^2} + 1 = \frac{2c^*}{y}, \qquad x^2 + (y - c^*)^2 = c^{*2}.$$

20. $(x^2 - 1)/y + y = 2c$ by algebra; $2x/y - [(x^2 - 1)/y^2 - 1]y' = 0$. Now replace y' by $-1/y'$ and multiply by y^2/x^2:

$$\frac{2yy'}{x} - \frac{y^2}{x^2} + 1 - \frac{1}{x^2} = \frac{d}{dx}\left(\frac{y^2}{x}\right) + 1 - \frac{1}{x^2} = 0.$$

Integration now gives

$$\frac{y^2}{x} + x + \frac{1}{x} = -2c^*.$$

Multiply by x to get the desired final formula

$$(x + c^*)^2 + y^2 = c^{*2} - 1.$$

22. $4x + 2yy' = 0, \quad y' = -2x/y, \quad y' = y/2x, \quad y = c^*\sqrt{x}$

24. $dv = 0$ gives $dy/dx = -v_x/v_y$ and this must equal u_y/u_x (see Prob. 23).

Differentiating $u = e^x \cos y$ and using the first Cauchy–Riemann equation, $u_x = v_y$, we obtain

$$v_y = u_x = e^x \cos y.$$

By integration with respect to y,

$$v = e^x \sin y + k(x),$$

where the "constant" of integration $k = k(x)$ depends on x because we are dealing with *partial* derivatives. From this and the second Cauchy–Riemann equation,

$$v_x = e^x \sin y + k'(x) = -u_y = + e^x \sin y.$$

Hence we must have $k'(x) = 0$, and $k = \tilde{c} = const.$ *Answer:* $e^x \sin y = c^*$.

26. TEAM PROJECT. (a) The point is that the student should learn to summarize the essential facts in a given more detailed presentation.

(b) Differentiating the given equation with respect to x and solving the result algebraically for y', we obtain the differential equation of the given curves

$$y' = -\frac{b^2 x}{a^2 y}.$$

This involves the constant $k = b^2/a^2$; hence we are dealing with infinitely many families, each corresponding to some value of k. The differential equation of the orthogonal trajectories is

$$y' = \frac{a^2 y}{b^2 x}.$$

By separation of variables and integration,

$$\ln |y| = \frac{a^2}{b^2} \ln |x| + \tilde{c}$$

and by taking exponentials,

$$y = c^* x^{a^2/b^2}.$$

We see that a^2/b^2 has substantial influence on the form of the trajectories. For $a^2 = b^2$ we have circles and obtain straight lines as trajectories. $a^2/b^2 = 2$ gives quadratic parabolas. For larger integer values of a^2/b^2 we obtain parabolas of higher order. Intuitively, the "flatter" the ellipses are, the more rapidly must the trajectories increase to have orthogonality.

(c) For hyperbolas we have a minus sign in the given formula. This produces a plus sign in the differential equation for the curves (instead of the minus we had) and a minus sign in the differential equation of the trajectories,

$$y' = -\frac{a^2 y}{b^2 x}.$$

By separation of variables and integration we obtain

$$y = c^* x^{-a^2/b^2},$$

with a minus sign in the exponent. For $a^2/b^2 = 1$ we get hyperbolas and for higher values less familiar curves.

(d) The problem set contains various cases that lead to other families of curves that can be handled easily.

SECTION 1.9. Existence and Uniqueness of Solutions. Picard Iteration, page 52

Purpose. To give the student at least some impression of the theory that would occupy a central position in a more theoretical course on a higher level.
Short Courses. This section can be omitted.

Comment on Iteration Methods

Iteration methods were used rather early in history, but it was Picard who made them popular. They are well suited for the computer because of their modest storage demands and usually short programs in which the same loop or loops are used many times, with different data. Since integration is generally no difficulty for a CAS, Picard's method has gained popularity during the past two decades.

SOLUTIONS TO PROBLEM SET 1.9, page 58

2. General solution $y = cx^4$, so that $y(0) = 0$ does not specify c and we have infinitely many solutions, $f(x, y) = 4y/x$ is not defined when $x = 0$. Note that in Prob. 1 we had no solutions; hence both cases, nonexistence or nonuniqueness, may occur.

4. Separating variables and integrating, we get

$$\frac{dy}{y} = \frac{2x-2}{x^2-2x}\,dx, \qquad \ln|y| = \ln|x^2 - 2x| + \tilde{c},$$

and by taking exponentials

$$y = c(x^2 - 2x) = cx(x-2).$$

From this we can see the *answers:*

(a) No solution if $y(0) = k \neq 0$ or $y(2) = k \neq 0$.

(b) Infinitely many solutions if $y(0) = 0$ or $y(2) = 0$.

(c) A unique solution satisfying $y(x_0) = y_0$ if $x_0 \neq 0$ and $x_0 \neq 2$. There are no contradictions to Theorems 1 and 2 because

$$f(x, y) = \frac{2x-2}{x^2-2x}$$

is not defined when $x = 0$ or 2.

6. $y \equiv 0$, $y = ce^{x^2/2}$ $(c > 0)$, $y = ce^{-x^2/2}$ $(c < 0)$. Thus the solutions in the upper half-plane increase very rapidly as $|x|$ increases, whereas in the lower half-plane they are bell-shaped curves and approach zero as $|x| \to \infty$. They are obtained by noting that $y' = xy$ if $y \geqq 0$ and $y' = -xy$ if $y \leqq 0$.

8. The smallest K is $K = (b+1)^2$, and $b/(b+1)^2$ is maximum when $b = 1$, the value is 1/4. Hence $\alpha = 1/4$. The solution is $y = 1/(2-x)$.

10. PROJECT. (a) The student should get an understanding of the "intermediate" position of a Lipschitz condition between continuity and (partial) differentiability.

(b) It suffices to consider the sine term. The validity of a Lipschitz condition follows from (12) in Appendix A3.1 and the calculation

$$|\sin y_2 - \sin y_1| = 2\left|\sin\frac{y_2 - y_1}{2}\right|\left|\cos\frac{y_2 + y_1}{2}\right| \leqq 2\,\frac{|y_2 - y_1|}{2}\cdot 1 = |y_2 - y_1|.$$

The nonexistence of $\partial f/\partial y$ can be seen from the curve of $|\sin x|$, which has a 90° cusp at 0; formally, if $x = 0$, then

$$\frac{|\sin(x + \Delta x)| - |\sin x|}{\Delta x} = \frac{|\sin \Delta x|}{\Delta x} \quad \to \quad \begin{cases} 1 & \text{if } \Delta x > 0 \\ -1 & \text{if } \Delta x < 0. \end{cases}$$

(c) Here the student should realize that the linear equation is basically simpler than the nonlinear one. The calculation is straightforward because we have

$$f(x, y) = r(x) - p(x)y$$

and this implies that

(A) $$f(x, y_2) - f(x, y_1) = -p(x)(y_2 - y_1).$$

This becomes a Lipschitz condition if we note that the continuity of $p(x)$ for $|x - x_0| \leqq a$ implies that $p(x)$ is bounded, say $|p(x)| \leqq M$ for all these x. Taking absolute values on both sides of (A) now gives

$$|f(x, y_2) - f(x, y_1)| \leqq M|y_2 - y_1|.$$

12. $y_n = \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^{n+1}}{(n+1)!}, \quad y = e^x - x - 1$

14. $y_0 = 1$, $y_1 = 1 + x$, $y_2 = 1 + x + x^2 + \frac{1}{3}x^3$, etc.; exact $y = 1/(1 - x)$

16. $y = (x - 1)^2$, $y \equiv 0$. The general solution is $y = (x + c)^2$. Picard iterations for this equation and other initial values are not suitable either. The student may give it a try for $y(1) = 1$, etc.

18. The solution is $y = x^3$. The Picard iterates are linear combinations of powers of $\ln x$,

$$1, \quad 1 + 3\ln x, \quad 1 + 3\ln x + \tfrac{9}{2}(\ln x)^2, \quad 1 + 3\ln x + \tfrac{9}{2}(\ln x)^2 + \tfrac{9}{2}(\ln x)^3,$$

etc.

20. CAS PROJECT. (b) The Maclaurin series is

$$y = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1\cdot 3\cdot 5\cdots(2n+1)}.$$

Picard's method gives the terms one after another, undisturbed by any error terms that change from step to step. The initial value problem is

$$y' = xy + 1, \qquad y(0) = 0.$$

This linear differential equation is solved as explained in Sec. 1.6.

(c) $y' = y$ would be a good candidate to begin with. It is perhaps a good idea to assume the initial choice in the form $y_0 + a$; then $a = 0$ corresponds to the choice in the text, and we see how the expressions in a are involved in the various approximations. The conjecture is true for any choice of a constant (or even of a continuous function of x).

SOLUTIONS TO CHAPTER 1 REVIEW, page 59

16. This Bernoulli equation (a Verhulst equation if $b < 0$) can be reduced to linear form, as shown in Example 5 of Sec. 1.6 (except for the notations). The general solution is (see (9) in Sec. 1.6)

$$y = \frac{1}{ce^{-ax} - b/a}.$$

18. We separate variables and integrate,

$$\frac{dy}{y^2+1} = -\frac{dx}{x^2+1}, \qquad \text{arc tan } y + \text{arc tan } x = \tilde{c}.$$

We now take the tangent on both sides and use the addition formula for the tangent (formula (16) in Appendix A3.1). This gives the *answer*.

$$\tan(\text{arc tan } y + \text{arc tan } x) = \frac{\tan(\text{arc tan } y) + \tan(\text{arc tan } x)}{1 - \tan(\text{arc tan } y)\tan(\text{arc tan } x)}$$

$$= \frac{y + x}{1 - xy} = c.$$

20. Exact equation, solvable almost by inspection,

$$e^{x^2}\cosh y + x = c.$$

22. $y/x = u$, $y = ux$, $y' = xu' + u$ substituted gives

$$x(xu' + u) = xu + x^2 \sec u.$$

xu drops out on both sides. Dividing by x^2, we get

$$u' = \sec u, \qquad \cos u\, du = dx, \qquad \sin u = x + c.$$

Answer: $y = xu = x \text{ arc sin}(x + c)$.

24. The equation is not exact. Theorem 1 in Sec. 1.5 gives an integrating factor $F = e^{x^2}$. Multiplying the equation by this factor, we see that it can be written

$$d(e^{x^2}\tan y) = 0. \qquad \textit{Answer: } e^{x^2}\tan y = c.$$

26. By separating variables, integrating, and simplifying we get

$$\frac{dy}{\sqrt{1 - y^2}} = dx, \qquad \text{arc sin } y = x + c, \qquad y = \sin(x + c),$$

the general solution. From this and the initial condition we obtain the *answer* $y = \sin(x + \frac{1}{4}\pi)$.

28. The general solution of this linear differential equation is obtained as explained in Sec. 1.6,

$$y = e^{-2x^2}\left(\int e^{2x^2}e^{-2x^2}\,dx + c\right) = (x + c)e^{-2x^2}.$$

From this and the initial condition $y(0) = -4$ we have $c = -4$. *Answer:*

$$y = (x - 4)e^{-2x^2}.$$

30. The exactness test gives $e^y = e^y$, so that the differential equation is exact. We have $u_y = xe^y$ from the equation. By integration,

$$u = xe^y + k(x)$$

By differentiation with respect to x and comparing with the coefficient function of dx in the equation, we get

$$u_x = e^y + k' = e^y + 2x; \qquad \text{thus} \qquad k' = 2x, \qquad k = x^2.$$

This gives the general solution

$$u = xe^y + x^2 = c.$$

The initial condition $y(2) = 0$ gives $2 \cdot 1 + 4 = 6$. *Answer:*

$$xe^y + x^2 = 6.$$

32. Theorem 1 in Sec. 1.5 gives the integrating factor $F = 1/x^2$. We thus obtain the exact equation

$$\frac{1}{x} \sinh y \, dy - \frac{1}{x^2} \cosh y \, dx = 0.$$

By inspection or systematically by integration (as explained in Sec. 1.5), we obtain

$$d\left(\frac{1}{x} \cosh y\right) = 0; \qquad \text{thus,} \qquad \frac{1}{x} \cosh y = c.$$

From this and the initial condition we get $\frac{1}{3} \cdot 1 = c$. *Answer:*

$$\cosh y = \tfrac{1}{3}x.$$

34. To solve this Bernoulli equation we set $u = y^{-2}$. Then $y = u^{-1/2}$, $y' = -\frac{1}{2}u^{-3/2}u'$. Substitution into the given differential equation gives

$$-\tfrac{1}{2}u^{-3/2}u' + \tfrac{1}{2}u^{-1/2} = u^{-3/2}.$$

We now multiply by $-2u^{3/2}$, obtaining

$$u' - u = -2. \qquad \text{General solution:} \qquad u = ce^x + 2.$$

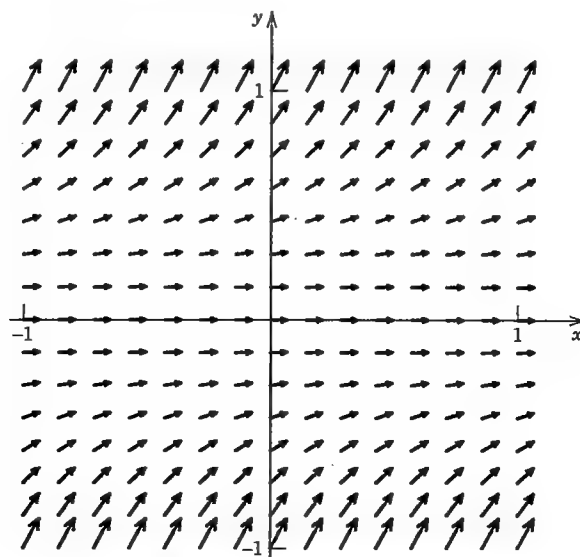
Hence

$$y = u^{-1/2} = \frac{1}{\sqrt{ce^x + 2}}.$$

From this and the initial condition $y(0) = 1$ we get $c = -1$. *Answer:*

$$y = \frac{1}{\sqrt{2 - e^x}}.$$

36. The student should gain confidence in the method by working simple equations that permit a comparison with exact solutions. This includes the choice of a suitable region to be plotted, using trial and error. *Solution:* $y = ce^{-x^2}$ (bell-shaped curves).

38. $y = 1/(c - 2x)$. See the figure, which shows the tangent directions of these hyperbolas.

Chapter 1 Review. Problem 38

40. The given curves can be written $x^3y = c$. By differentiation and simplification we get the differential equation of the given curves,

$$x^3y' + 3x^2y = 0, \qquad y' = -3y/x.$$

Hence the differential equation of the orthogonal trajectories is

$$y' = x/3y.$$

By separating variables we get as the general solution the family of orthogonal trajectories

$$y = \sqrt{x^2/3 + c^*}.$$

42. We square the given representation and differentiate the result,

$$y^2 = 2 \ln |x| + c, \qquad 2yy' = 2/x.$$

Hence the differential equation of the orthogonal trajectories is $y' = -xy$. We separate variables, integrate, and then take exponentials,

$$\frac{dy}{y} = -x\,dx, \qquad \ln |y| = c - x^2/2, \qquad y = c^*e^{-x^2/2}.$$

These are the orthogonal trajectories. This agrees with Prob. 41, where we went in the opposite direction.

44. Exact: $y = 1 + e^{-x}$. Iterates:

$$2, \qquad 2 - x, \qquad 2 - x + \tfrac{1}{2}x^2, \qquad 2 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3, \qquad \text{etc.}$$

46. By Newton's law of cooling, since the surrounding temperature is 100°C and the initial temperature of the metal is $T(0) = 20$, we first obtain

$$T(t) = 100 - 80e^{kt}.$$

k can be determined from the condition that $T(1) = 51.5$; that is,

$$T(1) = 100 - 80e^k = 51.5,$$

so that $k = \ln(48.5/80) = -0.500$. With this value of k we can now find the time at which the metal has the temperature 99.9°C,

$$99.9 = 100 - 80e^{-0.5t}, \qquad 0.1 = 80e^{-0.5t}, \qquad t = \frac{\ln 800}{0.5} = 13.4.$$

Answer: The temperature of the metal has practically reached that of the boiling water after 13.4 min.

48. We get 10 amperes from the 48-volt battery by choosing $R = 48/10 = 4.8$ [ohms]. Then $L = 0.007$ henry follows from the condition

$$I = 10(1 - e^{-Rt/L}) = 9.99.$$

Here we have used the initial condition $I(0) = 0$.

50. We proceed as in Sec. 1.4. The time rate of change $y' = dy/dt$ equals the inflow of salt minus the outflow per minute,

$$y' = 20 - \frac{20}{500}y.$$

The initial condition is $y(0) = 80$. This gives the particular solution

$$y = 500 - 420e^{-0.04t}.$$

The limiting value is 500 lb; 95% are 475 lb, so that we get the condition

$$500 - 420e^{-0.04t} = 475,$$

from which we can determine

$$t = 25 \ln \frac{420}{25} = 70.5 \text{ [min]};$$

so it will take a little over an hour.

52. This is Example 1 in Sec. 1.8 with the given curves and the trajectories interchanged. It also shows how these kinds of curves and orthogonal trajectories may occur in physics. From $x^2 + 2y^2 = c$ we obtain the differential equation $y' = -x/2y$. Hence the differential equation of the orthogonal trajectories is

$$y' = \frac{2y}{x}. \qquad \textit{Solution:} \quad y = c^*x^2.$$

54. The equation is separable,

$$\frac{dy}{(y-a)(y-b)} = k\,dt.$$

We now use partial fractions,

$$\frac{1}{(y-a)(y-b)} = \frac{1}{b-a}\left(\frac{1}{y-b} - \frac{1}{y-a}\right).$$

By integration and multiplication by $b - a$,

$$\ln(y-b) - \ln(y-a) = (kt + \tilde{c})(b-a).$$

Taking exponentials now gives

$$\frac{y-b}{y-a} = ce^{(b-a)kt}.$$

We can solve this algebraically for y. Denoting the function on the right by f, we obtain

$$y = \frac{b - af}{1 - f}.$$

CHAPTER 2: Linear Differential Equations of Second and Higher Order

Major Changes

The old Chap. 3 on higher order linear differential equations has been absorbed into Chap. 2 (Secs. 2.13–2.15). The main emphasis is on second-order differential equations. By combining these two chapters, trivial duplication is avoided, so that the entire presentation has become more streamlined.

SECTION 2.1. Homogeneous Linear Equations of Second Order, page 64

Purpose. To extend the basic concepts from first-order to second-order equations and to present the basic properties of linear equations.

Comment on the Standard Form (1)

The form (1), with 1 as the coefficient of y'', is practical, because if one starts from

$$f(x)y'' + g(x)y' + h(x)y = \tilde{r}(x),$$

one usually considers the equation in an interval I in which $f(x)$ is nowhere zero, so that in I one can divide by $f(x)$ and obtain an equation of the form (1). Points at which $f(x) = 0$ require a special study, which we present in Chap. 4.

Main Content, Important Concepts

Linear and nonlinear equations

Homogeneous linear equations (Secs. 2.1–2.7)

Nonhomogeneous linear equations (follow in Secs. 2.8–2.12, 2.15)

Superposition principle for homogeneous equations

General solution, basis, linear independence

Particular solution, initial value problem (2), (5)

Reduction to first order (Probs. 1–16)

Comment on the Four Equations Near the Beginning

These are for illustration, not for solution, but should a student ask, answers are that the first will be solved by methods in Secs. 2.9 and 2.10, the second is a Legendre equation (Sec. 4.3), the third has $y = x^{-1/2}$ as a solution, and the fourth is solved in Prob. 16.

Comment on Footnote 4

In 1760, Lagrange gave the first methodical treatment of the calculus of variations. The book mentioned in the footnote includes all major contributions of others in the field and made him the founder of analytical mechanics.

SOLUTIONS TO PROBLEM SET 2.1, page 71

2. $y'' = \dfrac{dy'}{dx} = \dfrac{dy'}{dy}\dfrac{dy}{dx} = \dfrac{dz}{dy}z$

4. $z = y'$, $2xz' = 3z$. Separation of variables and integration gives

$$\frac{dz}{z} = \frac{3}{2x}\,dx, \qquad \ln|z| = \frac{3}{2}\ln|x| + \tilde{c}, \qquad z = cx^{3/2}.$$

Integrating once more, we have

$$y = \int z\,dx = c_1x^{5/2} + c_2.$$

6. $p = 2/x$ (divide the equation by x to get it in standard form, with 1 as the coefficient of y''). Hence in (9),

$$e^{-\int p\,dx} = e^{\int 2/x\,dx} = \frac{1}{x^2}.$$

This gives from (9)

$$U = \frac{x^2}{\sin^2 x}\cdot\frac{1}{x^2} = \frac{1}{\sin^2 x}.$$

The integral of this is $-\cot x$, and thus

$$y_2 = uy_1 = -\frac{\cos x}{\sin x}\,\frac{\sin x}{x} = -\frac{\cos x}{x}.$$

8. $xz' + z = 0$, $\dfrac{dz}{z} = -\dfrac{dx}{x}$, $\ln|z| = -\ln|x| + \tilde{c}$, $z = \dfrac{c}{x}$, so that we obtain the answer

$$y = \int z\,dx = c\ln|x| + c_2.$$

10. $z = \dfrac{dy}{dx}$, $\dfrac{dz}{dy}z + \left(1 + \dfrac{1}{y}\right)z^2 = 0$, divide by z, separate variables, and integrate:

$$\frac{dz}{z} = -\left(1 + \frac{1}{y}\right)dy, \qquad \ln|z| = -y - \ln|y| + \tilde{c}.$$

Take exponentials, separate again, and integrate:

$$\frac{dy}{dx} = z = \frac{c}{y}e^{-y}, \qquad ye^y\,dy = c\,dx, \qquad \int ye^y\,dy = cx + c_2.$$

Evaluation of the integral gives the *answer* $(y - 1)e^y = c_1x + c_2$.

12. The standard form is

$$y'' - \frac{2x}{1 - x^2}y' + \frac{2}{1 - x^2}y = 0.$$

Hence in (9) we have

$$-\int p\,dx = \int \frac{2x}{1 - x^2}\,dx = -\ln(1 - x^2) = \ln\frac{1}{1 - x^2}.$$

This gives, in terms of partial fractions,

$$U = \frac{1}{x^2}\cdot\frac{1}{1 - x^2} = \frac{1}{x^2} + \frac{1/2}{x + 1} - \frac{1/2}{x - 1}.$$

By integration we get the *answer*

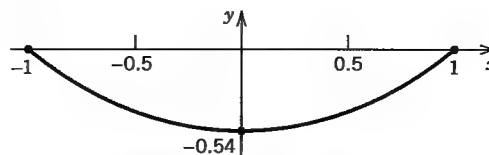
$$y_2 = y_1u = y_1\int U\,dx = -1 + \tfrac{1}{2}x\ln\frac{x + 1}{x - 1}.$$

The equation is **Legendre's equation** with parameter $n = 1$ (which, of course, need not be mentioned to the student), and the solution is essentially a Legendre function. Similarly, the equation in Prob. 11 is **Bessel's equation** with parameter $\frac{1}{2}$ (a case in which Bessel functions of the first kind reduce to sine and cosine (divided by $\sqrt{x}$)).

14. $y'' = y'$, $y(0) = 2$, $y'(0) = 2$, $y = 2e^t$, $y(6) = y'(6) = 807$

16. $z' = (1 + z^2)^{1/2}$, $(1 + z^2)^{-1/2}\,dz = dx$, $\sinh^{-1} z = x + c_1$. From this, $z = \sinh(x + c_1)$, $y = \cosh(x + c_1) + c_2$. From the boundary conditions $y(1) = 0$, $y(-1) = 0$ we get

$$\cosh(1 + c_1) + c_2 = 0 = \cosh(-1 + c_1) + c_2.$$

Hence $c_1 = 0$ and then $c_2 = -\cosh 1$. The *answer* is (see the figure) $y = \cosh x - \cosh 1$.

Section 2.1. Problem 16

18. Double root $(\lambda + 1)^2 = 0$, $y = (c_0 + c_1 x)e^{-x}$, $y(0) = c_0 = 1$, $y'(0) = -c_0 + c_1 = 0$, $c_1 = c_0 = 1$. *Answer:* $y = (1 + x)e^{-x}$.

Doing more such problems before the discussion of the (rather simple) solution method in the next section may scare students rather than really help them.

SECTION 2.2. Second-Order Homogeneous Equations with Constant Coefficients, page 72

Purpose. To show that constant-coefficient equations can be solved by algebra, namely, by solving the quadratic characteristic equation (3), which may have:

(Case I) Real distinct roots

(Case II) A real double root ("critical case")

(Case III) Complex conjugate roots (see Sec. 2.3 for details)

SOLUTIONS TO PROBLEM SET 2.2, page 75

2. $(c_1 + c_2 x)e^{-1.6x}$ **4.** $c_1 e^{\sqrt{8}x} + c_2 e^{-\sqrt{8}x}$ **6.** $c_1 e^{\pi x/4} + c_2 e^{-\pi x/4}$

8. $c_1 e^{-x} + c_2 e^{0.4x}$ **10.** $6e^{2x} + 4e^{-3x}$ **12.** $3e^{-x}$

14. $e^{-5x/2} - e^{5x/2}$ **16.** $[(k + 1)e^{kx} + (k - 1)e^{-kx}]/2k$

18. Linearly independent **20.** Linearly independent **22.** Linearly independent

24. Linearly dependent because $x|x| = x^2$ for nonnegative x

26. Linearly dependent because $\sin 2x = 2 \sin x \cos x$

28. Proportionality on I implies proportionality on J. No, proportionality on J does not imply proportionality on I. Probs. 24 and 25 illustrate this.

30. **TEAM PROJECT.** **(a)** $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = \lambda^2 + a\lambda + b$. Comparing coefficients gives $a = -(\lambda_1 + \lambda_2)$, $b = \lambda_1\lambda_2$.

(b) $y'' + ay' = 0$. (i) $c_1e^{-ax} + c_2e^{0x} = c_1e^{-ax} + c_2$. (ii) $z' + az = 0$, where $z = y'$, $z = ce^{-ax}$ and the second term comes in by integration, $y = \int z\,dx = c_1e^{-ax} + c_2$.

(d) $e^{(k+m)x}$ and e^{kx} satisfy $y'' - (2k + m)y' + k(k + m)y = 0$, by the coefficient formulas in part (a). By the superposition principle, another solution is

$$\frac{e^{(k+m)x} - e^{kx}}{m}.$$

We now let $m \to 0$. This becomes 0/0, and by l'Hôpital's rule (differentiation of numerator and denominator separately with respect to m, not x!) we obtain

$$xe^{kx}/1 = xe^{kx}.$$

The differential equation becomes $y'' - 2ky' + k^2y = 0$. The characteristic equation is

$$\lambda^2 - 2k\lambda + k^2 = (\lambda - k)^2 = 0$$

and has a double root. Since $a = -2k$, we get $k = -a/2$, as expected.

SECTION 2.3. Case of Complex Roots. Complex Exponential Function, page 76

Purpose. To discuss the remaining complex Case III, which gives undamped (harmonic) oscillations (if $c = 0$) or damped oscillations, first obtained in complex form, but convertible to the real form (9) by the superposition principle.

Main Content, Important Concepts

Real general solution (10) in Case III (a damped oscillation)

Euler formula (5) [resulting from the definition (7) of e^z]

Comment on How to Avoid Working in Complex

The average engineering student will profit from working a little with complex numbers. But if one has reasons for avoiding complex numbers here, one may apply the method of eliminating the first derivative from the equation, that is, substitute $y = uv$ and determine v so that the equation for u does not contain u'. For v this gives

$$2v' + av = 0. \qquad \text{A solution is} \qquad v = e^{-ax/2}.$$

With this v, the equation for u takes the form

$$u'' + (b - \tfrac{1}{4}a^2)u = 0$$

and can be solved by remembering from calculus that $\cos \tilde{\omega}x$ and $\sin \tilde{\omega}x$ reproduce under two differentiations, multiplied by $-\tilde{\omega}^2$. This gives (10), where

$$\tilde{\omega} = \sqrt{b - \tfrac{1}{4}a^2}.$$

Of course, the present approach can be used to handle all three cases. In particular, $u'' = 0$ in Case II gives $u = c_1 + c_2x$ at once.

Comment on Boundary Value Problems and Initial Value Problems

In usual courses on differential equations, initial value problems are generally given more space and weight than boundary value problems. Some reasons are that initial value problems have the following advantages:

1. They do not have the somewhat awkward nonuniqueness explained in Example 4.
2. Initial conditions are more suitable when a higher order equation is converted to a first-order system, as is usually done in existence and uniqueness theory.

For a first-order equation the two concepts formally coincide, but it seems a bit illogical to speak of a boundary value problem because a single point (at which the condition is given) does not bound any interval; it is not the "boundary" of anything, so the situation that suggested the name "boundary value problem" is not given in this case.

SOLUTIONS TO PROBLEM SET 2.3, page 80

2. $A \cos 2\pi x + B \sin 2\pi x$

4. $e^{-kx}(A \cos 2x + B \sin 2x)$

6. I, $c_1 e^{3x} + c_2 e^{-4x}$

8. III, $e^{-2x}(A \cos \omega x + B \sin \omega x)$

10. III, $e^{x\sqrt{2}}\left(A \cos \dfrac{x}{\sqrt{2}} + B \sin \dfrac{x}{\sqrt{2}}\right)$

12. III, $e^{-kx}\left(A \cos \dfrac{x}{k} + B \sin \dfrac{x}{k}\right)$

14. $-0.5e^{-2x} \cos \dfrac{x}{2}$

16. $e^{-0.2x}\left(\cos \dfrac{x}{2} - 2 \sin \dfrac{x}{2}\right)$

18. $e^{x}(-2 \cos 2\pi x + 3 \sin 2\pi x)$

20. $y = \cosh 5x$ by inspection. Systematically, we first get

$$y = c_1 e^{-5x} + c_2 e^{5x}.$$

From the boundary conditions,

$$\begin{aligned} y(-2) &= c_1 e^{10} + c_2 e^{-10} = \cosh 10 \\ y(2) &= c_1 e^{-10} + c_2 e^{10} = \cosh 10. \end{aligned}$$

By elimination or by Cramer's rule, $c_1 = c_2 = \frac{1}{2}$, in agreement with the result by inspection.

22. $y = c_1 e^{-x/3} + c_2 e^{3x}$. From the boundary conditions,

(a) $$y(-3) = c_1 e + c_2 e^{-9} = 1$$

(b) $$y(3) = c_1 e^{-1} + c_2 e^{9} = e^{-2}.$$

(a) minus e^2 times (b) gives $c_2 = 0$. Then $c_1 = 1/e$ from (a). *Answer:* $y = e^{-x/3-1}$

24. **PROJECT.** The purpose is twofold: (i) Students should learn to look at results carefully before rushing on to the next project or problem, and (ii) graphs may show various interesting facts not obvious from formulas. They may also give quantitative impressions (e.g., in this case, how rapidly the exponential function decreases). Since the tangent at the extrema is horizontal, whereas at the points of contact the tangent has a negative slope (for positive y) or a positive slope (for negative y), it is clear without calculation that these points cannot coincide with extrema, but must come after them (at larger x's). For the harmonic motion the inflection points lie on the axis, for reasons of symmetry. For a damped oscillation, one might guess that they are alternatingly at positive and negative y-values, shifted from the intersection points

slightly to smaller x-values. Some calculations are as follows.

$$y = e^{-0.1x}\sin 2x$$

$$y' = e^{-0.1x}(-0.1\sin 2x + 2\cos 2x) = 0, \qquad \tan 2x = 20,$$

thus $x = 0.760\,419$, etc.,

$$y'' = e^{-0.1x}(-0.4\cos 2x - 3.99\sin 2x) = 0, \qquad \tan 2x = -\frac{0.4}{3.99},$$

thus $x = -0.049\,958 + \pi/2$, etc.

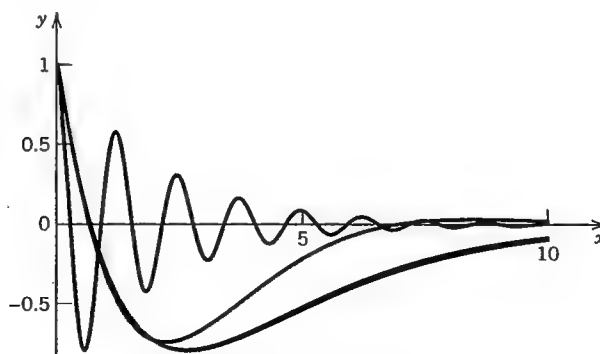
26. CAS PROJECT. **(a)** $\omega^2 = b - \frac{1}{4}a^2$ by the definition of ω. Thus $b = \frac{1}{4} + \omega^2$ because $a = 1$.

(b) The approach is rapid. The figure shows the solutions for $\omega = 5, 0.5, 0.1, 0.01, \cdots, 0.000\,001$.

(c) $y = e^{-x/2}\left(\cos\omega x - \dfrac{3}{2\omega}\sin\omega x\right)$. Applying l'Hôpital's rule to the second term (differentiating numerator and denominator separately with respect to ω, not x), we get the limit

$$e^{-x/2}(\cos\omega x - \tfrac{3}{2}(\cos\omega x)x)|_{\omega=0} = (1 - \tfrac{3}{2}x)e^{-x/2},$$

as it should be.

(d) Change $y'(0)$ to 0 or to a positive value.

Section 2.3. CAS Project 26

SECTION 2.4. Differential Operators. *Optional,* page 81

Purpose. To take a short look at the operational calculus of second-order differential operators with constant coefficients, which parallels and confirms our discussion of differential equations with constant coefficients.

SOLUTIONS TO PROBLEM SET 2.4, page 83

2. $-12x^2 - 10x + 4, 0, 2\sin 2x - 6\cos 2x$

4. $25(2 + 5x + 2\cos 5x), 20(1 + 5x)e^{5x}, 0$

6. $(c_1 + c_2x)e^{-x/3}$ **8.** $c_1e^{0.2x} + c_2e^{-0.2x}$ **10.** $c_1e^{\pi x} + c_2e^{-\pi^2 x}$

12. $c_1e^{-x/2} + c_2$

SECTION 2.5. Modeling: Free Oscillations (Mass–Spring System), page 83

Purpose. To present a main application of second-order constant-coefficient equations

$$my'' + cy' + ky = 0$$

resulting as models of motions of a mass m on an elastic spring of modulus k (> 0) under linear damping c ($\geqq 0$) by applying Newton's second law and Hooke's law. These are free motions (no driving force). Forced motions follow in Sec. 2.11.

Main Content, Important Concepts

Restoring force ky, damping force cy', force of inertia my''

No damping, harmonic oscillations (4), natural frequency $\omega_0/2\pi$

Overdamping, critical damping, nonoscillatory motions (7), (8)

Underdamping, damped oscillations (10)

SOLUTIONS TO PROBLEM SET 2.5, page 90

2. $W = 20$ and $s_0 = 2$ gives $k = W/s_0 = 10$ by Hooke's law. Thus

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{\sqrt{k/(W/g)}}{2\pi} = \frac{\sqrt{10/(20/980)}}{2\pi} = 3.52 \text{ [Hz]}.$$

From this we get the period $1/f = 0.284$ [sec].

4. No, because the frequency is independent of initial conditions; it only depends on k/m.

6. By Hooke's law, $F_1 = k_1 = 8$ stretches spring S_1 by 8, and $F_2 = k_2 = 12$ stretches spring S_2 by 12. Hence the unknown k of the combination of the springs stretches S_1 by $k/k_1 = k/8$ and S_2 by $k/k_2 = k/12$. And k is such that the sum of these stretches equals 1, because k is the force that corresponds to the stretch 1 of the combination. Thus

$$\frac{k}{k_1} + \frac{k}{k_2} = 1, \qquad \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{k}. \qquad \textit{Answer: } k = 4.8.$$

8. $my'' = -\pi \cdot 0.3^2 y\gamma$, where $\pi \cdot 0.3^2 y$ is the volume of water displaced when the buoy is depressed y meters from its equilibrium position, and $\gamma = 9800$ nt is the weight of water per cubic meter. Thus $y'' + {\omega_0}^2 y = 0$, where ${\omega_0}^2 = \pi \cdot 0.3^2\gamma/m$ and the period is $2\pi/\omega_0 = 2$; hence

$$m = \pi \cdot 0.3^2\gamma/{\omega_0}^2 = 0.3^2\gamma/\pi = 281,$$

$$W = mg = 281 \cdot 9.80 = 2754 \text{ [nt]} \quad \text{(about 620 lb)}.$$

10. TEAM PROJECT. (a) $mL\theta'' = -mg \sin\theta \approx -mg\theta$ (the tangential component of $w = mg$), $\theta'' + {\omega_0}^2\theta = 0$, ${\omega_0}^2 = g/L$. *Answer:* $\sqrt{g/L}/2\pi$

(b) By (a), the frequency is

$$\frac{1}{2\pi}\sqrt{\frac{g}{L}} = \frac{1}{2\pi}\sqrt{\frac{9.80}{1}} = 0.498,$$

so it takes about 2 sec to complete 1 cycle. *Answer:* It ticks about 30 times per minute.

(c) $W = ks_0 = 8$. Now $s_0 = 1$ because the system has its equilibrium position 1 cm below the horizontal line. Also, $m = W/g$, so that

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{W/s_0}{W/g}} = \sqrt{g} = \sqrt{980} = 31.3,$$

and we get the general solution

$$y = A \cos 31.3t + B \sin 31.3t.$$

The initial conditions give $y(0) = A = 0$ and $y'(0) = 31.3B = 10$. Hence $B = 0.319$ and the *answer* is

$$y = 0.319 \sin 31.3t \text{ [cm]}.$$

(d) $\theta(t) = 0.5235 \cos 3.7t + 0.0943 \sin 3.7t$ [rad]

12. $y = 0$ gives $c_1 = -c_2 e^{-2\beta t}$, which has at most one solution because the exponential function is monotone.

14. Equating the derivative of (8) to zero, we get

$$y' = (-\alpha c_1 - \alpha c_2 t + c_2)e^{-\alpha t} = 0$$

and from this the solution

$$t = t_0 = \frac{c_2 - \alpha c_1}{\alpha c_2} = \frac{1}{\alpha} - \frac{c_1}{c_2}$$

which is positive if $1/\alpha > c_1/c_2$. This is the condition.

16. From (10) and $y' = 0$ we obtain $\tan(\omega^* t - \delta) = -\alpha/\omega^* = const$ and consecutive solutions of this equation have the constant distance π/ω^*.

18. If a maximum is at t_0, the next is at $t_1 = t_0 + 2\pi/\omega^*$. Since the sine and cosine in (10) have period $2\pi/\omega^*$, the ratio is

$$\exp(-\alpha t_0)/\exp(-\alpha t_1) = \exp(2\pi\alpha/\omega^*).$$

$\Delta = 2\pi$; $\tan t = -1$ gives $3\pi/4$ (min), $7\pi/4$ (max), etc.

20. CAS PROJECT. **(a)** Cases I, II, III appear, along with their typical solution curves, no matter what k/m is or $y(0)$, etc.

(b) The first step is to see that Case II corresponds to $c = 2$. Then one can choose other values by experimentation. In Fig. 51 the values of c (omitted on purpose; the student should choose!) are 0 and 0.1 for the oscillating curves and 1, 1.5, 2, 3 for the others (from below to above).

(c) This addresses a general issue that also arises in problems involving heating and cooling, mixing, electrical vibrations, etc. One is generally surprised how quickly certain states are reached practically when the theoretical time is infinite.

(d) $y(t) = e^{-ct/2}(A \cos \omega^* t + B \sin \omega^* t)$, $\omega^* = \frac{1}{2}\sqrt{4 - c^2}$. From the initial conditions, $A = 1$, $B = c/\sqrt{4 - c^2}$. In $y'(t)$ the cosine terms drop out, and $\sin \omega^* t = 0$ gives as the smallest positive solution $t = t_2 = 2\pi/\sqrt{4 - c^2} = \pi/\omega^*$. There $y(t)$ has a horizontal tangent and touches $y = -0.01$ when $y(t_2) = -0.01$ and stays within the limits in (11) because it oscillates between $\pm e^{-ct_2/2}$. Thus we get c from $y(t_2) = -e^{-ct_2/2} = -0.01$ as $c = 1.65$, approximately.

(e) The main difference is that Case II gives $y(t) = (1 - t)e^{-t}$, which is negative for $t > 1$. The experiments with the curves are as before.

SECTION 2.6. Euler–Cauchy Equation, page 93

Purpose. Algebraic solution of the Euler–Cauchy equation, which appears in certain applications (see our Example 4) and which we shall need again in Sec. 4.4 as the simplest equation to which the Frobenius method applies. We have three cases; this is similar to the situation for constant-coefficient equations, to which the Euler–Cauchy equation can be transformed (Prob. 20); however, this fact is of theoretical rather than of practical interest.

Comment on Footnote 9

Euler worked in St. Petersburg 1727–1741 and 1766–1783 and in Berlin 1741–1766. He investigated Euler's constant (Sec. 4.6) first in 1734, used Euler's formula (Secs. 2.3, 12.6, 12.7) since 1740, introduced integrating factors (Sec. 1.5) in 1764, and studied conformal mappings (Sec. 12.5) since 1770. His main influence on the development of mathematics and mathematical physics resulted from his textbooks, in particular from his famous *Introductio in analysin infinitorum* (1748), in which he also introduced many of the modern notations (for trigonometric functions, etc.). Euler was the central figure of the mathematical activity of the 18th century. His collected works are still incomplete, although some seventy volumes have already been published.

Cauchy worked in Paris, except during 1830–1838 when he was in Turin and Prague. In his two fundamental works, *Cours d'Analyse* (1821) and *Résumé des leçons données à l'École royale polytechnique* (vol. 1, 1823), he introduced more rigorous methods in calculus, based on an exactly defined limit concept; this also includes his convergence principle (Sec. 14.1). He also was the first to give existence proofs in differential equations. He initiated complex analysis; we discuss his main contributions to this field in Secs. 12.4, 13.2–13.4, and 14.2. His famous integral theorem (Sec. 13.2) was published in 1825, his paper on complex power series and their radius of convergence (Sec. 14.2) in 1831.

SOLUTIONS TO PROBLEM SET 2.6, page 96

2. I, $c_1x^2 + c_2x^3$

4. I, $c_1/x + c_2$. This can also be solved by reduction to first order and separation of variables.

6. III, $x[A\cos(\ln x) + B\sin(\ln x)]$

8. II, $c_1 + c_2 \ln x$. Also solvable by reduction and separation.

10. I, $c_1x^{-0.2} + c_2x^{0.5}$

12. II, $(c_1 + c_2 \ln x)x^{0.6}$

14. General solution: $c_1x + c_2x^2$. *Answer:* $2x - \frac{1}{2}x^2$

16. General solution: $A\cos(3\ln x) + B\sin(3\ln x)$. *Answer:* $2\cos(3\ln x)$

18. General solution: $(c_1 + c_2 \ln x)/x$. *Answer:* $(3 - \ln x)/x$

20. $x = e^t$, $t = \ln x$. The chain rule gives

$$y' = \dot{y}t' = \dot{y}/x, \qquad y'' = \ddot{y}/x^2 - \dot{y}/x^2,$$

where the dots denote derivatives with respect to t. By substitution into (1) we obtain

$$x^2y'' + axy' + by = x^2\left(\frac{\ddot{y}}{x^2} - \frac{\dot{y}}{x^2}\right) + ax\frac{\dot{y}}{x} + by = \ddot{y} + (a-1)\dot{y} + by = 0.$$

The characteristic equation of the new equation is

$$\lambda^2 + (a - 1)\lambda + b = 0.$$

It is of the form (3). Its roots are in Case I

$$e^{m_1 t} = (e^t)^{m_1} = x^{m_1}, \qquad e^{m_2 t} = x^{m_2}$$

etc., so that we can obtain the solution of the Euler–Cauchy equation from those of the new equation. Also, in Case III the transformation into real form in Sec. 2.3 carries over into that in this section.

SECTION 2.7. Existence and Uniqueness Theory. Wronskian, page 97

Purpose. To explain the theory of existence of solutions of equations with variable coefficients in standard form (that is, with y'' as the first term, not, say, $f(x)y''$)

$$y'' + p(x)y' + q(x)y = 0$$

and of their uniqueness if initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1$$

are imposed. Of course, no such theory was needed in the last sections on equations for which we were able to write down all solutions explicitly.

Main Content

Continuity of coefficients suffices for existence and uniqueness.

Linear independence if and only if the Wronskian is not zero.

General solution exists and includes all solutions.

Comment on Wronskian

For $n = 2$, where linear independence and dependence can be seen immediately, the Wronskian serves primarily as a tool in our proofs; the practical value of the independence criterion will appear for higher n in Sec. 2.13.

SOLUTIONS TO PROBLEM SET 2.7, page 100

2. $W = \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} = e^x$

4. $W = (m_2 - m_1)x^{m_1+m_2-1}$

6. $W = \begin{vmatrix} e^{\lambda x} & xe^{\lambda x} \\ \lambda e^{\lambda x} & (1 + \lambda x)e^{\lambda x} \end{vmatrix} = \begin{vmatrix} 1 & x \\ \lambda & 1 + \lambda x \end{vmatrix} e^{2\lambda x} = e^{2\lambda x}$

8. We use the abbreviations $c = \cos \omega x$, $s = \sin \omega x$. Then

$$W = \begin{vmatrix} e^{-x}c & e^{-x}s \\ e^{-x}(-c - \omega s) & e^{-x}(-s + \omega c) \end{vmatrix} = e^{-2x}\begin{vmatrix} c & s \\ -c - \omega s & -s + \omega c \end{vmatrix} = \omega e^{-2x}.$$

10. $x^2y'' + xy' - 25y = 0$, $W = -10/x$

12. $y'' - 4y = 0$, $W = \begin{vmatrix} c & s \\ 2s & 2c \end{vmatrix} = 2$, where $c = \cosh 2x$, $s = \sinh 2x$

14. $y'' + 2y' = 0$, $W = -2e^{-2x}$

16. $x^2y'' + xy' + y = 0$, $W = \begin{vmatrix} c & s \\ -s/x & c/x \end{vmatrix} = \dfrac{1}{x}$,

where $c = \cos(\ln x)$, $s = \sin(\ln x)$.

18. TEAM PROJECT. (a) Suppose that y_1 and y_2 are zero at some point x_0 in I. Then the first row of their Wronskian is zero at x_0. This implies linear dependence of y_1 and y_2 by Theorem 2.

(b) At a maximum or minimum the first derivative is zero; if this happens for two solutions y_1 and y_2 at the same point, the second row of the Wronskian is zero, so $W = 0$ at that point. This implies linear dependence by Theorem 2.

(c) By direct calculation,

$$W(z_1, z_2) = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix} = \begin{vmatrix} a_{11}y_1 + a_{12}y_2 & a_{21}y_1 + a_{22}y_2 \\ a_{11}y_1' + a_{12}y_2' & a_{21}y_1' + a_{22}y_2' \end{vmatrix}$$

$$= (a_{11}y_1 + a_{12}y_2)(a_{21}y_1' + a_{22}y_2') - (a_{11}y_1' + a_{12}y_2')(a_{21}y_1 + a_{22}y_2).$$

Multiplying out, we see that four of the eight terms cancel in pairs (the terms in y_1y_1' and y_2y_2'). The remaining terms can be written

$$(a_{11}a_{22} - a_{12}a_{21})(y_1y_2' - y_2y_1') = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} W(y_1, y_2).$$

From this the conclusion follows. Note that in this calculation we need not refer to the familiar rule for multiplying determinants (which some students may not know).

(d) $\det [a_{jk}] = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

(e) $c_1e^x + c_2e^{-x} = \tilde{c}_1 \cosh x + \tilde{c}_2 \sinh x = \frac{1}{2}\tilde{c}_1(e^x + e^{-x}) + \frac{1}{2}\tilde{c}_2(e^x - e^{-x})$ gives $c_1 = \frac{1}{2}(\tilde{c}_1 + \tilde{c}_2)$, $c_2 = \frac{1}{2}(\tilde{c}_1 - \tilde{c}_2)$.

SECTION 2.8. Nonhomogeneous Equations, page 101

Purpose. We show that for getting a general solution y of a nonhomogeneous linear equation we must find a general solution y_h of the corresponding homogeneous equation and then—this is our new task—any particular solution y_p of the nonhomogeneous equation,

$$y = y_h + y_p.$$

Main Content, Important Concepts

General solution, particular solution

Continuity of p, q, r suffices for existence and uniqueness.

General solution exists and includes all solutions.

(Solution methods follow in Secs. 2.9, 2.10.)

SOLUTIONS TO PROBLEM SET 2.8, page 103

2. The general solution of the homogeneous equation is $c_1e^{-x} + c_2e^x$. Hence as the solution of the nonhomogeneous equation we first obtain

$$y = y_h + y_p = c_1e^{-x} + c_2e^x + e^{-3x} - 3e^x.$$

We see that the last term is a solution of the homogeneous equation, and we can absorb it into the general solution of the latter, so that we simply have

$$y = c_1e^{-x} + \tilde{c}_2e^x + e^{-3x},$$

the same answer as in Prob. 1, except for the notation. Of course, the point of the problem is that two particular solutions of the nonhomogeneous equation can differ at most by a solution of the homogeneous equation; in the present case, this is $-3e^x$.

4. $y = e^x(A \cos 2x + B \sin 2x) + x^3$. Whereas in Prob. 3 we have just one term on the right side of the equation but many terms in y_p, here we have the opposite situation where the right side of the equation has many terms but y_p is simple.

6. $y = (c_1 + c_2x)e^{2x} - \frac{1}{2}e^x \sin x$

8. $y = c_1e^{x/4} + c_2e^{x/2} + \frac{1}{5}e^{-x} + e^x$. Perhaps the student should express y_p in terms of $\cosh x$ and $\sinh x$, to see the analogy to the expression $A \cos x + B \sin x$ in other equations.

10. $y = 0.4e^x + 0.6e^{-x} - \cos x$. From this form of the answer we recognize the form of the general solution $y_h = c_1e^x + c_2e^{-x}$ (which may not always be the case). It is important for the student to understand that y_p ***will satisfy the initial conditions only in very rare cases***—practically never—and that further work is necessary for solving the initial value problem.

12. $y = 1.8 \cos 2x + \sin 2x + 3x \cos 2x$. The right side of the equation is a solution of the homogeneous equation and produces the form of y_p involving the factor x. This will be discussed systematically in the next section.

14. $y = 4x - 2x^2 + 3e^x$. The first two terms result from the general solution of the homogeneous equation $c_1x + c_2x^2$.

16. TEAM PROJECT. **(a)** 1. Find a general solution of the homogeneous equation.

2. Find any particular solution y_p of (1). (It is quite unlikely that y_p automatically satisfies the initial conditions.)

3. Determine values of the arbitrary constants in (3) from the initial conditions.

(b) The difference of the two solutions must be a solution of the homogeneous equation.

(c) As in (b).

(d) Of course, because y_p does not depend on the choice of that general solution y_h or $\tilde{y}_h$.

(e) The usual method for the Euler–Cauchy equation gives the general solution $c_1x + c_2x^2$ of the homogeneous equation, hence $y = c_1x + c_2x^2 + 3e^x$ for the nonhomogeneous equation. From this, $y(0) = 3$ (note that any other $y(0)$ would result in no solution!). Now $y' = c_1 + 2c_2x + 3e^x$, $y'(0) = c_1 + 3 = 7$, hence $c_1 = 4$, whereas c_2 remains arbitrary. The reason is that the coefficients of the equation in standard form

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0$$

become infinite as $x \to 0$.

SECTION 2.9. Solution by Undetermined Coefficients, page 104

Purpose. To discuss a special method for particular solutions of constant-coefficient equations with special right side $r(x)$. This method is simpler than that in Sec. 2.10 and should be used whenever it applies. Rules (A), (B), and (C) tell us what to do in practice.

Comment on Table 2.1

It is clear that the table could be extended by the inclusion of products of polynomials times cosine or sine and other cases of limited practical value. Also, $\alpha = 0$ in the last pair of lines gives the previous two lines, which we have listed separately because of their practical importance.

SOLUTIONS TO PROBLEM SET 2.9, page 107

The request to show each step should prevent students from simply letting the CAS produce the final answer.

2. $y = c_1e^{-x} + c_2e^{x} + xe^{x} + 2e^{2x}$. An important point is that the Modification Rule applies only to one of the two exponential terms. The Sum Rule is also used.

4. $y = c_1e^{-x} + c_2e^{2x} + xe^{2x}$; an application of the Modification Rule for a simple root $\lambda = 2$.

6. $y = c_1e^{-3x} + c_2e^{-x/3} + 3x - 10 + \frac{1}{2}\sin x$; an application of the Sum Rule. Note that $9x$ causes an x-term and a constant term in the solution. The cosine term would usually cause a cosine and a sine term, so here we get less than expected.

8. $y = (c_1 + c_2x)e^{-3x} + e^{-x}(6\cos x + 8\sin x)$. In this problem we go slightly beyond the lines in Table 2.1, which does not contain products of trigonometric times exponential functions. However, the method is the same in principle and should encourage students to attempt more independent work. On the other hand, we did not include other such problems, whose practical value is not very great.

10. $y = c_1e^{-x/2} + c_2e^{3x/2} - 2e^{-2x} + \frac{42}{5}e^{2x}$. Students should perhaps be asked to express the solutions (the last two terms) in terms of $\cosh 2x$ and $\sinh 2x$, to see the analogy to expressions $a\cos x + b\sin x$ in other differential equations.

12. $y = c_1e^{3x} + c_2e^{-6x} + \frac{1}{4}e^{-3x} + \frac{1}{2}xe^{3x}$; Modification Rule. Here e^{3x} is "hidden" in $\sinh 3x$ on the right, whereas the other term in $\sinh 3x$ does not call for the Modification Rule but produces $\frac{1}{4}e^{-3x}$ without an extra factor x.

14. $y = e^{2x}(A\cos 4x + B\sin 4x) + 4\cos x + 19\sin x$

16. $y = e^{3x}(\cos 2x - \sin 2x) + e^{3x}$. Be sure students do not get confused: the Modification Rule is *not* needed.

18. $y = \cos 3x + x\sin 3x$. The first term results from the general solution $c_1\cos 3x + c_2\sin 3x$ of the homogeneous equation. Initial conditions $y(0) = 1, y'(0) = 0$ (or conversely) appear in various theoretical considerations.

20. $y = x^2e^{1.4x}$. This is an application of the Modification Rule in the case of the double root 1.4. A general solution of the nonhomogeneous equation is $y = (c_1 + c_2x + x^2)e^{1.4x}$. One should emphasize that the initial conditions $y(0) = 0$, $y'(0) = 0$ would imply $y \equiv 0$ only in the homogeneous case. Also, $c_1 = 0, c_2 = 0$ is an exception, caused by $y_p(0) = 0, y_p'(0) = 0$, where $y_p = x^2e^{1.4x}$.

22. $2e^{-0.5x}\cos 3x + e^{-x} + 4$. The general solution $e^{-0.5x}(A\cos 3x + B\sin 3x)$ of the homogeneous equation contributes the first term of the solution.

24. TEAM PROJECT. Begin with simple cases. Find the form of y_p (with undetermined coefficients) by repeated differentiation. For example, xe^x will require $y_p = (a + bx)e^x$, etc. Applications may occur occasionally. For instance, $e^{-kx} \cos \omega x$ (with $x = t$ = time) could represent a time-decreasing driving force.

SECTION 2.10. Solution by Variation of Parameters, page 108

Purpose. To discuss the general method for particular solutions, which applies in any case, but may often lead to difficulties in integration (which we by and large have avoided in our problems, as the subsequent answers show).

Comments

The equation must be in ***standard form***, with 1 as the coefficient of y''—students tend to forget that.

Here we do need the Wronskian, in contrast to Sec. 2.7 where we could get away without it.

SOLUTIONS TO PROBLEM SET 2.10, page 111

2. $y_1 = \cos 3x, \quad y_2 = \sin 3x, \quad W = 3.$ Hence in (2),

$$\int \frac{y_2 r}{W}\, dx = \int \frac{\sin 3x \sec 3x}{3}\, dx = -\frac{1}{9} \ln |\cos 3x|$$

$$\int \frac{y_1 r}{W}\, dx = \int \frac{\cos 3x \sec 3x}{3}\, dx = \frac{1}{3} x.$$

Answer:

$$y = A \cos 3x + B \sin 3x + \tfrac{1}{9}(\cos 3x) \ln |\cos 3x| + \tfrac{1}{3} x \sin 3x.$$

4. $y_1 = \cos 3x, \quad y_2 = \sin 3x, \quad W = 3, r = \csc 3x.$ Hence in (2),

$$\int \frac{y_2 r}{W}\, dx = \frac{1}{3} \int \frac{\sin 3x}{\sin 3x}\, dx = \frac{x}{3}$$

$$\int \frac{y_1 r}{W}\, dx = \frac{1}{3} \int \frac{\cos 3x}{\sin 3x}\, dx = \frac{1}{9} \ln |\sin 3x|.$$

Answer:

$$A \cos 3x + B \sin 3x - \frac{x}{3} \cos 3x + \frac{1}{9} (\sin 3x) \ln |\sin 3x|$$

6. $y_1 = e^{2x} \cos x, \quad y_2 = e^{2x} \sin x, \quad W = e^{4x}.$ Hence in (2),

$$\int \frac{y_2 r}{W}\, dx = \int \frac{(e^{2x} \sin x) e^{2x} / \sin x}{e^{4x}}\, dx = x$$

$$\int \frac{y_1 r}{W}\, dx = \int \frac{(e^{2x} \cos x) e^{2x} / \sin x}{e^{4x}}\, dx = \ln |\sin x|.$$

Answer:

$$y = [A \cos x + B \sin x - x \cos x + (\sin x) \ln |\sin x|] e^{2x}.$$

8. $y_1 = e^{-3x}, \quad y_2 = xe^{-3x}, \quad W = e^{-6x}.$ Hence in (2),

$$\int \frac{y_2 r}{W}\,dx = \int \frac{(xe^{-3x})16e^{-3x}/(x^2+1)}{e^{-6x}}\,dx = \int \frac{16x}{x^2+1}\,dx = 8 \ln (x^2+1)$$

$$\int \frac{y_1 r}{W}\,dx = \int \frac{(e^{-3x})16e^{-3x}/(x^2+1)}{e^{-6x}}\,dx = \int \frac{16}{x^2+1}\,dx = 16 \arctan x.$$

Answer:

$$y = (c_1 + c_2 x)e^{-3x} + 8[-\ln (x^2+1) + 2x \arctan x]e^{-3x}.$$

10. $y_1 = e^{-x}\cos x$, $y_2 = e^{-x}\sin x$, $W = e^{-2x}$. Hence in (2),

$$\int \frac{y_2 r}{W}\,dx = \int \frac{(e^{-x}\sin x)4e^{-x}/\cos^3 x}{e^{-2x}}\,dx = \frac{2}{\cos^2 x}$$

$$\int \frac{y_1 r}{W}\,dx = \int \frac{(e^{-x}\cos x)4e^{-x}/\cos^3 x}{e^{-2x}}\,dx = 4 \tan x.$$

This gives the particular solution

$$e^{-x}\left[-(\cos x)\frac{2}{\cos^2 x} + 4(\sin x)\tan x\right] = e^{-x}\left(\frac{-2 + 4\sin^2 x}{\cos x}\right)$$
$$= e^{-x}[-2(\cos 2x)/\cos x].$$

Answer:

$$y = e^{-x}[A\cos x + B\sin x - 2(\cos 2x)/\cos x].$$

12. $y_1 = 1$, $y_2 = x^2$, $W = 2x$. Divide the given equation by x to get it in standard form and from it,

$$r = (3+x)x^2e^x/x = (3+x)xe^x.$$

Hence in (2),

$$\int \frac{y_2 r}{W}\,dx = \int \frac{x^2(3+x)xe^x}{2x}\,dx = \frac{1}{2}\int (x^3 + 3x^2)e^x\,dx = \frac{1}{2}x^3e^x$$

$$\int \frac{y_1 r}{W}\,dx = \int \frac{1\cdot(3+x)xe^x}{2x}\,dx = \frac{1}{2}\int (x+3)e^x\,dx = \tfrac{1}{2}(x+2)e^x.$$

Substitution into (2) shows that the terms in x^3 drop out and the *answer* is

$$y = c_1 + c_2x^2 + x^2e^x.$$

14. $y_1 = x^3$, $y_2 = x^2$, $W = -x^4$. From the standard form we get

$$r = (7x^4 \sin x)/x^2 = 7x^2 \sin x.$$

Hence in (2),

$$\int \frac{y_2 r}{W}\,dx = \int \frac{x^2(7x^2\sin x)}{-x^4}\,dx = 7\cos x$$

$$\int \frac{y_1 r}{W}\,dx = \int \frac{x^3(7x^2\sin x)}{-x^4}\,dx$$
$$= -7\int x\sin x\,dx = 7x\cos x - 7\sin x.$$

This gives the particular solution

$$-7x^3\cos x + x^2(7x\cos x - 7\sin x) = -7x^2\sin x.$$

Answer:

$$y = c_1x^2 + c_2x^3 - 7x^2\sin x.$$

16. $y_1 = x, \quad y_2 = 1/x, \quad W = -2/x$. From (2),

$$y_p = -x\int -\frac{x}{2}\frac{1}{x}\frac{1}{x^4}\,dx + \frac{1}{x}\int -\frac{x}{2}\,x\,\frac{1}{x^4}\,dx = \frac{1}{3x^2}\,.$$

Answer:

$$y = c_1 x + c_2 x^{-1} + \frac{1}{3x^2}\,.$$

18. TEAM PROJECT. (a) $y_1 = e^{-3x}, \quad y_2 = e^{-x}, \quad W = 2e^{-4x}, \ r = 65\cos 2x$. From (2),

$$\begin{aligned} y_p &= -e^{-3x}\int \frac{e^{-x}65\cos 2x}{2e^{-4x}}\,dx + e^{-x}\int \frac{e^{-3x}65\cos 2x}{2e^{-4x}}\,dx \\ &= \frac{65}{2}\left(-e^{-3x}\int e^{3x}\cos 2x\,dx + e^{-x}\int e^{x}\cos 2x\,dx\right) \\ &= \frac{65}{2}\left(-e^{-3x}\frac{1}{13}\,e^{3x}(3\cos 2x + 2\sin 2x) + e^{-x}\frac{1}{5}\,e^{x}(\cos 2x + 2\sin 2x)\right) \\ &= -\cos 2x + 8\sin 2x. \end{aligned}$$

Answer:

$$y = c_1 e^{-3x} + c_2 e^{-x} - \cos 2x + 8\sin 2x.$$

This was much more work than that for undetermined coefficients.

(b) We can treat x^2 on the right by undetermined coefficients, obtaining the contribution $x^2 + 4x + 6$ to the solution. We could treat it by the other method, but we would have to evaluate additional integrals of an exponential function times a power of x. We treat the other part, $35x^{3/2}e^x$, by the method of this section, calling the resulting function y_{p1}. We need $y_1 = e^x$, $y_2 = xe^x$, $W = e^{2x}$. From this and (2),

$$\begin{aligned} y_{p1} &= -e^{x}\int \frac{xe^{x}}{e^{2x}}\,35x^{3/2}e^{x}\,dx + xe^{x}\int \frac{e^{x}}{e^{2x}}\,35x^{3/2}e^{x}\,dx \\ &= 35\left(-e^{x}\int x^{5/2}\,dx + xe^{x}\int x^{3/2}\,dx\right) = 4e^{x}x^{7/2}. \end{aligned}$$

Complete answer:

$$y = (c_1 + c_2 x)e^{x} + 4e^{x}x^{7/2} + x^2 + 4x + 6.$$

(c) If the right side is a power of x, say, $r = r_0 x^k$, then substitution of $y_p = Cx^k$ gives

$$x^2y'' + axy' + by = (k(k-1) + ak + b)Cx^k = r_0 x^k.$$

This can be solved for C. To explore further possibilities, one may work "backwards"; that is, assume a solution, substitute it on the left, and see what form one gets as a right side.

SECTION 2.11. Modeling: Forced Oscillations. Resonance, page 111

Purpose. To extend Sec. 2.5 from free to forced vibrations by adding an input (a driving force, here assumed to be sinusoidal). Mathematically, we go from a homogeneous to a nonhomogeneous equation, which we solve by undetermined coefficients.

New Features

Resonance (11) $y = At \sin \omega_0 t$ in the undamped case

Beats (12) $y = B(\cos \omega t - \cos \omega_0 t)$ if input frequency close to natural

Large amplitude if (15′) $\omega^2 = \omega_0{}^2 - c^2/2m^2$ (Fig. 60)

Phase lag between input and output

SOLUTIONS TO PROBLEM SET 2.11, page 117

2. $y_p = 1.5 \cos 3t + \sin 3t$

4. $y_p = \frac{1}{10} \cos t - \frac{1}{90} \cos 3t + \frac{1}{5} \sin t + \frac{1}{45} \sin 3t$

6. $y_p = \frac{1}{2}(\cos t + \sin t)$

8. $y = A \cos \sqrt{3}t + B \sin \sqrt{3}t + 4 \cos 0.5t$. We have no sine term in y_p because of the absence of y' in the equation. This is typical.

10. $y = (c_1 + c_2 t)e^{-3t} + 2 \sin t - \frac{3}{2} \cos t$

12. $y = c_1 e^{-t} + c_2 + t - \frac{1}{2} \cos t + \frac{1}{2} \sin t$. Note that the harmless term 1 on the right of the equation causes the unbounded term t in the solution.

14. $y = e^{-t}(\cos t + 2 \sin t) + 0.2 \cos t + 0.4 \sin t$. For $t = 5$ the exponential term has decreased to less than 1% of its original value; this *practically* marks the end of the transition.

16. $y = e^{-4t} \cos t + 26.8 \sin 0.5t - 6.4 \cos 0.5t$. At $t = 1.2$ the exponential term has decreased to less than 1% of its original value. This marks the end of the transition from a practical point of view. $t = 1.8$ is the time when that term has become less than 1/10 of a percent in absolute value.

18. WRITING PROJECT. Brevity should force the student to recognize what is important and what is marginal. It is useful to learn this in connection with short reports, articles, talks, etc.

20. CAS PROJECT. The choice of ω needs experimentation, inspecting the curves obtained and then making changes on a trial-and-error basis. It is interesting to see how in the case of beats the period gets longer and longer and the maximum amplitudes get larger and larger as ω approaches the resonance frequency.

SECTION 2.12. Modeling of Electric Circuits, page 118

Purpose. To discuss the current in the *RLC*-circuit with sinusoidal input $E_0 \sin \omega t$. **ATTENTION!** The right side in (1) is $E_0\omega \cos \omega t$, because of differentiation.

Main Content

Modeling by a simple extension of Sec. 1.7

Electrical–mechanical strictly quantitative analogy (Table 2.2)

Transient tending to harmonic steady-state current

SOLUTIONS TO PROBLEM SET 2.12, page 122

2. $\alpha = R/2L > 0$. If β is real, $\beta \leqq R/2L$ since $R^2 - 4L/C \leqq R^2$; hence $\lambda_1 = -\alpha + \beta < 0$ (and $\lambda_2 = -\alpha - \beta < 0$, of course). If β is imaginary, $I_h(t)$ represents a damped oscillation.

4. $10 \cos t + 20 \sin t$ because the general solution of the homogeneous equation approaches zero as $t \to \infty$.

6. 0

8. $E' = 1700 \cos 4t$, $I = e^{-2t}(A \cos 4t + B \sin 4t) + 5 \cos 4t + 20 \sin 4t$

10. The equation is

$$10I'' + 80I' + 250I = 2405 \cos 10t.$$

Using undetermined coefficients, we obtain the general solution

$$I = e^{-4t}(A \cos 3t + B \sin 3t) - \tfrac{3}{2} \cos 10t + \tfrac{8}{5} \sin 10t.$$

Hence $y(0) = A - \frac{3}{2} = 0$, $A = \frac{3}{2}$. From Eq. (1″) for the charge we see that $Q(0) = 0$ in the present case implies $I'(0) = 0$. By differentiating I and substituting $A = \frac{3}{2}$ we obtain from $I'(0) = 0$ the value $B = -\frac{10}{3}$. *Answer:*

$$I = e^{-4t}(\tfrac{3}{2} \cos 3t - \tfrac{10}{3} \sin 3t) - \tfrac{3}{2} \cos 10t + \tfrac{8}{5} \sin 10t.$$

12. The equation is

$$0.5I'' + 3I' + 12.5I = -60 \sin 5t.$$

It has the general solution

$$y = e^{-3t}(A \cos 4t + B \sin 4t) + 4 \cos 5t.$$

From (1″) for Q we obtain (similar to Example 1) $I'(0) = Q''(0) = 24$. From this and $I(0) = 0$ we obtain the *answer*

$$I = e^{-3t}(3 \sin 4t - 4 \cos 4t) + 4 \cos 5t.$$

14. The equation is

$$2I'' + \tfrac{1}{5} \cdot 10^5 I = 0, \qquad \text{thus} \qquad I'' + 10^4 I = 0.$$

A general solution is

$$I = A \cos 100t + B \sin 100t.$$

$I(0) = 0$ gives $A = 0$. Equation (1″) for the charge is

$$2Q'' + 2 \cdot 10^4 Q = 110.$$

It implies that

$$I'(0) = Q''(0) = 55$$

because $Q(0) = 0$. From $I = B \sin 100t$ we thus obtain

$$I' = 100\,B \cos 100t, \qquad I'(0) = 100B = 55, \qquad B = 0.55.$$

Answer:

$$I = 0.55 \sin 100t.$$

16. **(a)** By integration,

$$Q = c - 0.0055 \cos 100t \qquad \text{with} \qquad c = 0.0055 \text{ from } Q(0) = 0.$$

(b) $2Q'' + 2 \cdot 10^4 Q = 110$; a general solution is

$$Q = A \cos 100t + B \sin 100t + 110/(2 \cdot 10^4).$$

From this and the first initial condition, $Q(0) = A + 0.0055 = 0$. Hence $A = -0.0055$. The second initial condition $I(0) = Q'(0) = 0$ gives $B = 0$ because

$$I = Q' = -100A \sin 100t + 100B \cos 100t = 0.$$

Together we have

$$Q = 0.0055 - 0.0055 \cos 100t,$$

as in (a).

18. TEAM PROJECT. **(a)** The complex division trick is performed to make the denominator real,

$$\frac{a}{b} = \frac{a\bar{b}}{|b|^2}.$$

Before we multiply out and take the real part, the expression for I_p is

$$I_p = Ke^{i\omega t} = \frac{-E_0}{S^2 + R^2}(S + iR)(\cos \omega t + i \sin \omega t).$$

(c) Substitution of (11) and its derivatives into the present equation gives

$$(-1 + i + 3)Ke^{it} = 5e^{it}.$$

Solving for K, we obtain $K = 2 - i$. Hence the complex solution is

$$I_p = (2 - i)e^{it} = (2 - i)(\cos t + i \sin t).$$

The real part is

$$I_p = 2 \cos t + \sin t.$$

The student should verify that it satisfies the given real differential equation.

SECTION 2.13. Higher Order Linear Equations, page 124

Purpose. Extension of the basic concepts and theory in Secs. 2.1 and 2.7 to homogeneous linear differential equations of any order n. This shows that all the essential facts carry over practically without change. Linear independence, now more involved than for $n = 2$, causes the Wronskian to become indispensable (whereas for $n = 2$ it played a marginal role).

Main Content, Important Concepts

Superposition principle for the homogeneous equation (2)

General solution, basis, particular solution

General solution of (2) with continuous coefficients exists.

Existence and uniqueness of solution of initial value problem (2), (5)

Linear independence of solutions, Wronskian

General solution includes all solutions of (2).

SOLUTIONS TO PROBLEM SET 2.13, page 131

2. $W = -6e^{2x}$, $y = e^x - 3e^{2x}$

4. $W = 1$, $y = 12 + 3 \cos x$

6. $W = 4$, $y = e^x$. Note that another basis is e^x, e^{-x}, $\cos x$, $\sin x$.

8. $W = 18$, $y = \cos x + \frac{1}{2} \sin 2x$

10. Linearly independent. Point out that $\sin 2x = 2 \sin x \cos x$ is not a linear combination of $\cos x$ and $\sin x$.

12. Linearly dependent. This is an example where the use of a functional relation helps

to decide: $\ln (x^3) = 3 \ln x$.

14. Linearly dependent. The essential point is that the exponential functions have the right exponent occurring in the definition of $\sinh 3x$.

16. Linearly independent

18. Linearly dependent. This serves as a reminder that any set containing the zero function as an element is linearly dependent.

20. TEAM PROJECT. (a) (1) If $y_1 \equiv 0$, then (4) holds with any $k_1 \neq 0$ and the other k_j all zero.

(2) If S were linearly dependent on I, then (4) would hold with a $k_j \neq 0$ on I, hence also on J, contradicting the assumption. This also shows that linear dependence on I implies linear dependence on J. Linear independence on I implies no conclusion for J. Example: $x|x|$ and x^2 are linearly independent on $-1 < x < 1$ but linearly dependent on $0 < x < 1$.

(3) By assumption, $k_1y_1 + \cdots + k_py_p = 0$ with $k_1, \cdots, k_p$ not all zero. This implies (4) with $k_1, \cdots, k_p$ as before and $k_{p+1} = \cdots = k_n = 0$. In the other case T may be linearly dependent (or not). Example: Take any linearly independent S and let T be S and the zero function.

(b) If your functions are solutions of a homogeneous linear differential equation with continuous coefficients, then you can use the Wronskian. For other means, see the problems (for instance, the use of functional relations, evaluating (4) at several x's in the interval, etc.).

SECTION 2.14. Higher Order Homogeneous Equations, page 132

Purpose. Extension of the algebraic solution method for constant-coefficient equations from $n = 2$ (Secs. 2.2, 2.3) to any n, and discussion of the increased number of possible cases:

Real different roots

Complex simple roots

Real multiple roots

Complex multiple roots

Combinations of the preceding four basic cases

Explanation of these cases in terms of typical examples.

Comment on Numerical Work

In practical cases, one may have to use Newton's method or another method for computing (approximate values of) roots in Sec. 17.2.

SOLUTIONS TO PROBLEM SET 2.14, page 137

2. $(c_1 + c_2x + c_3x^2)e^{-3x}$ **4.** $(c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x$

6. $A_1 \cos x + A_2 \cos 2x + B_1 \sin x + B_2 \sin 2x$

8. $c_1e^{-2x} + (c_2 + c_3x)e^{x}$ **10.** $c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$

12. $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$. Hence a general solution is

$$y = (c_1 + c_2x + c_3x^2)e^{x}, \qquad \text{and} \qquad y(0) = c_1 = 2.$$

With this,

$$y' = (2 + c_2x + c_3x^2 + c_2 + 2c_3x)e^x$$

and

$$y'(0) = 2 + c_2 = 2, \qquad c_2 = 0.$$

With $c_2 = 0$, another differentiation gives

$$y'' = (2 + c_3x^2 + 2c_3x + 2c_3x + 2c_3)e^x$$

and

$$y''(0) = 2 + 2c_3 = 10, \qquad c_3 = 4.$$

Answer:

$$y = (2 + 4x^2)e^x.$$

14. The characteristic equation has the roots ± 1 and $\pm i$. Hence a general solution, its derivatives, and their values at $x = 0$, equated to the corresponding initial conditions, are as follows.
(a) $y = c_1e^x + c_2e^{-x} + A\cos x + B\sin x, \quad c_1 + c_2 + A = -1$
(b) $y' = c_1e^x - c_2e^{-x} - A\sin x + B\cos x, \quad c_1 - c_2 + B = 7$
(c) $y'' = c_1e^x + c_2e^{-x} - A\cos x - B\sin x, \quad c_1 + c_2 - A = -1$
(d) $y''' = c_1e^x - c_2e^{-x} + A\sin x - B\cos x, \quad c_1 - c_2 - B = 7.$
We obtain $A = 0$ from (a), (c); then $B = 0$ from (b), (d); then $c_1 = 3$ from (a), (b); and finally $c_2 = -4$ from (a). *Answer:*

$$y = 3e^x - 4e^{-x}.$$

16. The characteristic equation has the roots $\pm i$ and $\pm 3i$. Hence a general solution, its derivatives, and their values at $x = 0$, equated to the corresponding initial conditions, are as follows.
(a) $y = A_1\cos x + B_1\sin x + A_2\cos 3x + B_2\sin 3x, \quad A_1 + A_2 = 0$
(b) $y' = -A_1\sin x + B_1\cos x - 3A_2\sin 3x + 3B_2\cos 3x, \quad B_1 + 3B_2 = 0$
(c) $y'' = -A_1\cos x - B_1\sin x - 9A_2\cos 3x - 9B_2\sin 3x, \quad -A_1 - 9A_2 = 32$
(d) $y''' = A_1\sin x - B_1\cos x + 27A_2\sin 3x - 27B_2\cos 3x, \quad -B_1 - 27B_2 = 0.$
From (a) and (c) we obtain $A_2 = -4$, $A_1 = 4$. From (b) and (d) we obtain $B_2 = 0$, $B_1 = 0$. *Answer:*

$$y = 4\cos x - 4\cos 3x.$$

18. The characteristic equation has a triple root -2. Hence a general solution, its derivatives, and their values at $x = 0$, equated to the corresponding initial conditions, are as follows.
(a) $y = (c_1 + c_2x + c_3x^2)e^{-2x}, \quad y(0) = c_1 = 1$
(b) $y' = (-2 + c_2 - 2c_2x + 2c_3x - 2c_3x^2)e^{-2x}, \quad y'(0) = -2 + c_2 = -2, \quad c_2 = 0$
(c) $y'' = 2(2 + c_3 - 4c_3x + 2c_3x^2)e^{-2x}, \quad y''(0) = 4 + 2c_3 = 6, \quad c_3 = 1.$
Answer:

$$y = (1 + x^2)e^{-2x}.$$

20. PROJECT. (a) Divide the characteristic equation by $\lambda - \lambda_1$ if $y_1 = e^{\lambda_1 x}$ is known.

(b) The idea is the same as in Sec. 2.1.

(c) Here, as always, the first step is to produce the standard form, as the form under which the equation for z was derived. Division by x^3 gives

$$y''' - \frac{3}{x}y'' + \left(\frac{6}{x^2} - 1\right)y' - \left(\frac{6}{x^3} - \frac{1}{x}\right)y = 0.$$

With $y_1 = x$, $y_1' = 1$, $y_1'' = 0$, and the coefficients p_1 and p_2 from the standard equation, we obtain

$$xz'' + \left[3 + \left(-\frac{3}{x}\right)x\right]z' + \left[2\left(-\frac{3}{x}\right) \cdot 1 + \left(\frac{6}{x^2} - 1\right)x\right]z = 0.$$

Simplification gives

$$xz'' + \left(-\frac{6}{x} + \frac{6}{x} - x\right)z = x(z'' - z) = 0.$$

Hence

$$z = c_1 e^x + \tilde{c}_2 e^{-x}.$$

By integration we get the *answer*

$$y_2 = x \int z\,dx = (c_1 e^x + c_2 e^{-x} + c_3)x.$$

SECTION 2.15. Higher Order Nonhomogeneous Equations, page 138

Purpose. To show that the transition from $n = 2$ (Sec. 2.8) to general n introduces no new ideas, but generalizes all results and practical aspects in a straightforward fashion; this refers to existence, uniqueness, and the need for a particular solution y_p to get a general solution in the form

$$y = y_h + y_p.$$

SOLUTIONS TO PROBLEM SET 2.15, page 141

2. $y_1 = x^{-1}$, $y_2 = x$, $y_3 = x^2$, $W = 6x^{-1}$, $W_1 = x^2$, $W_2 = -3$, $W_3 = 2x^{-1}$. Furthermore, $r = \ln x$ because we have to divide the equation by x^3 to get it in standard form. From (7) we now obtain

$$\begin{aligned} y_p &= \frac{x^{-1}}{6}\int x^3 \ln x\,dx - \frac{x}{2}\int x \ln x\,dx + \frac{x^2}{3}\int \ln x\,dx \\ &= \frac{x^{-1}}{6}\left[\frac{x^4}{4}\ln x - \frac{x^4}{16}\right] - \frac{x}{2}\left[\frac{x^2}{2} - \frac{x^2}{4}\right] + \frac{x^2}{3}[x \ln x - x] \\ &= \frac{x^3}{8}\ln x - \frac{7}{32}x^3. \end{aligned}$$

Answer:

$$y = c_1 x^{-1} + c_2 x + c_3 x^2 + \tfrac{1}{8}x^3 \ln x - \tfrac{7}{32}x^3.$$

4. y_p is conveniently obtained by undetermined coefficients. *Answer:*

$$y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^x + 2x^3 - 3x^2 + 15x - 8.$$

6. $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3$. Hence a basis is

$$y_1 = e^{2x}, \qquad y_2 = xe^{2x}, \qquad y_3 = x^2 e^{2x}.$$

The derivatives are

$$y_1' = 2e^{2x}, \qquad y_2' = (2x + 1)e^{2x}, \qquad y_3' = (2x^2 + 2x)e^{2x}.$$

The second derivatives are

$$y_1'' = 4e^{2x}, \qquad y_2'' = (4x + 4)e^{2x}, \qquad y_3'' = (4x^4 + 8x + 2)e^{2x}.$$

From the Wronskian we can factor out e^{2x} from each of the three columns. Then

$$W = e^{6x} \begin{vmatrix} 1 & x & x^2 \\ 2 & 2x + 1 & 2x^2 + 2x \\ 4 & 4x + 4 & 4x^2 + 8x + 2 \end{vmatrix}$$

$$= e^{6x} \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 4 & 8x + 2 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 2x \\ 4 & 8x + 2 \end{vmatrix} = 2e^{6x}.$$

In (7) we further need

$$W_1 = e^{4x} \begin{vmatrix} 0 & x & x^2 \\ 0 & 1 & 2x \\ 1 & 4 & 8x + 2 \end{vmatrix} = e^{4x} \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = e^{4x}x^2,$$

where we have e^{4x} instead of e^{6x} because in the replacement of one column of W as explained in the text, we lose a factor e^{2x}. Furthermore,

$$W_2 = e^{4x} \begin{vmatrix} 1 & 0 & x^2 \\ 0 & 0 & 2x \\ 0 & 1 & 8x + 2 \end{vmatrix} = e^{4x} \begin{vmatrix} 0 & 2x \\ 1 & 2x + 2 \end{vmatrix} = -2e^{4x}x$$

and

$$W_3 = e^{4x} \begin{vmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{vmatrix} = e^{4x} \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} = e^{4x}.$$

With these values and

$$r = x^{1/2}e^{2x}$$

the integrals in (7) become

$$\int \frac{W_1 r}{W}\,dx = \int \frac{e^{4x}x^2x^{1/2}e^{2x}}{2e^{6x}}\,dx$$

$$= \frac{1}{2}\int x^{5/2}\,dx = \frac{1}{7}\,x^{7/2},$$

$$\int \frac{W_2 r}{W}\,dx = \int \frac{-2e^{4x}xx^{1/2}e^{2x}}{2e^{6x}}\,dx$$

$$= -\int x^{3/2}\,dx = -\frac{2}{5}\,x^{5/2},$$

$$\int \frac{W_3 r}{W}\,dx = \int \frac{e^{4x}x^{1/2}e^{2x}}{2e^{6x}}\,dx$$

$$= \frac{1}{2}\int x^{1/2}\,dx = \frac{1}{3}\,x^{3/2}.$$

From this and (7),

$$y_p = e^{2x}\frac{1}{7}x^{7/2} - xe^{2x}\frac{2}{5}x^{5/2} + x^2e^{2x}\frac{1}{3}x^{3/2}$$

$$= e^{2x}x^{7/2}\frac{1}{105}(15 - 42 + 35) = \frac{8}{105}e^{2x}x^{7/2}.$$

Answer:

$$y = \left[c_1 + c_2x + c_3x^2 + \tfrac{8}{105}x^{7/2}\right]e^{2x}.$$

8. $y_1 = x$, $y_2 = x^{1/2}$, $y_3 = x^{3/2}$, $W = -\frac{1}{4}$, $W_1 = x$, $W_2 = -\frac{1}{2}x^{3/2}$, $W_3 = -\frac{1}{2}x^{1/2}$, $r = x^{5/2}$ (divide by $4x^3$). From (7) we thus obtain

$$y_p = x\int(-4x)x^{5/2}\,dx + x^{1/2}\int 2x^{3/2}x^{5/2}\,dx + x^{3/2}\int 2x^{1/2}x^{5/2}\,dx$$

$$= -\tfrac{8}{9}x^{11/2} + \tfrac{2}{5}x^{11/2} + \tfrac{1}{2}x^{11/2}$$

$$= \tfrac{1}{90}x^{11/2}.$$

Answer:

$$y = c_1x + c_2x^{1/2} + c_3x^{3/2} + x^{11/2}/90.$$

10. $y_1 = x$, $y_2 = x\ln x$, $y_3 = x(\ln x)^2$, $W = 2$, $W_1 = x(\ln x)^2$, $W_2 = -2x\ln x$, $W_3 = x^2$, $r = 1/x$. *Answer:* $y = x^2 + x\ln x$

12. $y = \sin x + \sin 3x + 2\sinh x$

14. CAS PROJECT. The first equation has as a general solution

$$y = (c_1 + c_2x + c_3x^2)e^x + \tfrac{8}{105}e^x x^{7/2},$$

so in cases such as this, one could try

$$y = x^{1/2}(a_0 + a_1x + a_2x^2 + a_3x^3)e^x.$$

However, the equation alone does not show much, so another idea is needed. One could modify the right side systematically and see how the solution changes. The solution of the second suggested equation shows that the equation is not accessible by undetermined coefficients; its solution is (see Prob. 2)

$$y = c_1x^{-1} + c_2x + c_3x^2 + \tfrac{1}{8}x^3\ln x - \tfrac{7}{32}x^3.$$

And one could perhaps modify this equation, too, in an attempt to obtain a form of solution that might be suitable for undetermined coefficients.

SOLUTIONS TO CHAPTER 2 REVIEW, page 142

16. $y = e^{-3x}(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x)$

18. $y = c_1x^3 + c_2x^{-3}$

20. y_p is obtained by the method of undetermined coefficients. *Answer:*

$$y = c_1e^x + c_2e^{-x} + c_3e^{4x} - 5e^{2x}.$$

22. The particular solution $y_p = x^2e^{\pi x}$ is obtained by the method of undetermined coefficients. *Answer:*

$$y = (c_1 + c_2x)e^{\pi x} + x^2e^{\pi x}.$$

24. The particular solution $y_p = -\ln x\, e^{-2x}$ is obtained by the method of variation of parameters. *Answer:*

$$y = (c_1 + c_2 x - \ln x)e^{-2x}.$$

26. The particular solution $y_p = \frac{1}{2}x \sinh x - \frac{3}{4}\cosh x$ is obtained by the method of undetermined coefficients. *Answer:*

$$y = c_1 + c_2 e^{x} + c_3 e^{-x} + \tfrac{1}{2}x \sinh x - \tfrac{3}{4}\cosh x.$$

28. Applying the method of undetermined coefficients, we obtain as a general solution

$$y = e^{-x/2}(A \cos \tfrac{1}{2}\sqrt{3}\,x + B \sin \tfrac{1}{2}\sqrt{3}\,x) + \sin x - 3\cos 2x + 2 \sin 2x.$$

30. By variation of parameters we obtain the *answer*

$$y = (c_1 + c_2 x)e^{2x} + (x \ln x - x)e^{2x}.$$

32. The particular solution $y_p = 3\cos x + \sin x$ is obtained by the method of undetermined coefficients. *Answer:*

$$y = 3e^{x} - 5e^{2x} + 3\cos x + \sin x.$$

34. $y = e^{-2x}(\cos \omega x + \sin \omega x)$

36. The initial conditions are such that the general solution of the homogeneous equation does not contribute to the *answer*

$$y = y_p = 2\cos x + \sin x.$$

38. $y = (2 - x)e^{x}$

40. $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3$; hence a general solution of the homogeneous equation is

$$y_h = (c_1 + c_2 x + c_3 x^2)e^{-x}.$$

By the method of undetermined coefficients and from the initial conditions we get the *answer*

$$y = (1 + x^2)e^{-x} - 2\cos x - 2\sin x.$$

42. $I(t) = c_1 e^{-1999.87t} + c_2 e^{-0.125008t}$

44. $I(t) = 0.0833e^{-160t} - 0.3333e^{-40t} + 0.2500$ if $0 < t < 0.01$,
$I(t) = -0.3294e^{-160t} + 0.1639e^{-40t}$ if $t > 0.01$.

Note that since $E(t)$ is continuous at $t = 0.01$, and Q is always continuous (cannot change abruptly), I and I' are continuous at $t = 0.01$, whereas I'' has a jump -1600 since $10E'$ has this jump at $t = 0.01$.

46. The complex equation is

$$4I'' + 20I' + 2I = 10 \cdot 10e^{10it}.$$

Substituting $I = Ke^{10it}$ and its derivatives and dropping the factor e^{10it}, we obtain

$$[4(-100) + 20 \cdot 10i + 2]K = 100.$$

Solving algebraically for K, we get

$$K = \frac{100(-398 - 200i)}{(-398 + 200i)(-398 - 200i)} = \frac{-39800 - 20000i}{198404} = -0.2006 - 0.1008i.$$

Answer:

$$\operatorname{Re} Ke^{10it} = -0.2006 \cos 10t + 0.1008 \sin 10t.$$

48. The equation is

$$0.125y'' + 1.125y = \cos t - 4 \sin t;$$

thus,

$$y'' + 9y = 8 \cos t - 32 \sin t.$$

The solution satisfying the initial conditions is

$$y = -\cos 3t + \tfrac{4}{3} \sin 3t + \cos t - 4 \sin t,$$

as obtained by the method of undetermined coefficients.

The last two terms result from the driving force. In the first two terms, $\omega_0 = \sqrt{k/m} = 3$. This shows that resonance would occur if the driving force had the frequency $\omega/2\pi = 3/2\pi$.

50. $C^*(\omega)$ is given by (14), Sec. 2.11. The maximum is obtained by equating the derivative to zero; this gives (15) in Sec. 2.11, which for our numerical values becomes

$$16 = 2(24 - \omega^2),$$

so that $\omega = 4$. Eq. (16) in Sec. 2.11 then gives the maximum amplitude

$$C^*(\omega_{max}) = \frac{2 \cdot 1 \cdot 10}{4\sqrt{4 \cdot 1^2 \cdot 24 - 16}} = 0.5590.$$

To check this result, we determine the general solution, using the method of undetermined coefficients, finding

$$y(t) = e^{-2t}(A \cos 2\sqrt{5}t + B \sin 2\sqrt{5}t) + 0.25 \cos 4t + 0.5 \sin 4t,$$

and confirm the result by calculating the amplitude

$$\sqrt{0.25^2 + 0.5^2} = 0.5590.$$

CHAPTER 3 Systems of Differential Equations. Phase Plane, Qualitative Methods

Major Changes

This chapter has been completely rewritten, on the basis of suggestions by instructors who have taught from it and of my own recent experience of (once more!) teaching systems of differential equations. The main reason is that due to the increasing emphasis on linear algebra in our standard curricula, we can now expect that when students take a course on differential equations that includes material from Chap. 3, almost all of them have at least some working knowledge of 2×2 matrices.

Accordingly, Chap. 3 makes modest use of **2×2 matrices.** $n \times n$ matrices are mentioned only in passing and are immediately followed by illustrative examples of systems of two differential equations in two unknowns, involving 2×2 matrices only. Section 3.2 and the beginning of Sec. 3.3 are intended to give the student the impression that for first-order systems, one can develop a theory that is conceptually and structurally similar to that in Chap. 2 for a single differential equation. Hence if the instructor feels that the class might be disturbed by $n \times n$ matrices, omission of the latter and explanation of the material in terms of two differential equations in two unknowns will entail no disadvantage and will leave no gaps of understanding or skill.

To be completely on the safe side, Sec. 3.0 is included for reference, so that the student will have no need to search through Chap. 6 or 7 for a concept or fact needed in Chap. 3.

Basic throughout Chap. 3 is the **eigenvalue problem** (for 2×2 matrices), consisting first of the determination of the eigenvalues λ_1, λ_2 (not necessarily numerically distinct) as solutions of the characteristic equation, that is, the quadratic equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0,$$

and then an eigenvector corresponding to λ_1 with components x_1, x_2 from

$$(a_{11} - \lambda_1)x_1 + a_{12}x_2 = 0$$

and an eigenvector corresponding to λ_2 from

$$(a_{11} - \lambda_2)x_1 + a_{12}x_2 = 0.$$

It may be useful to emphasize early that eigenvectors are determined only up to a nonzero factor and that in the present context, normalization (in order to obtain unit vectors) is hardly of any advantage.

If there are students in the class who have not seen eigenvalues before (although the elementary theory of these problems does occur in every up-to-date introductory text on linear algebra), they should not have difficulties in readily grasping the meaning of these problems and their role in this chapter, simply because of the numerous examples and applications in Sec. 3.3 and in later sections.

Section 3.5 includes three famous applications, namely, the pendulum and van der Pol equations and the **Lotka–Volterra predator–prey population model.**

SECTION 3.0. Introduction: Vectors, Matrices, Eigenvalues, page 146

Purpose. This section is for reference and review only, the material being restricted to what is actually needed in this chapter, to make it self-contained.

Main Content

Matrices, vectors

Algebraic matrix operations

Differentiation of vectors

Eigenvalue problems for 2×2 matrices

Important Concepts and Facts

Matrix, column and row vector, multiplication

Linear independence

Eigenvalue, eigenvector, characteristic equation

Some Details on Content

Most of the material is explained in terms of 2×2 matrices, which play the major role in Chap. 3; indeed, $n \times n$ matrices for general n occur only briefly in Sec. 3.2 and at the beginning in Sec. 3.3. Hence the demand on the student in Chap. 3 will be very modest, and Sec. 3.0 is written accordingly.

In particular, eigenvalue problems presently lead to quadratic equations only, so that nothing needs to be said about difficulties encountered with 3×3 or larger matrices.

Example 1. Although the later sections include many eigenvalue problems, the complete solution of such a problem (the determination of the eigenvalues and corresponding eigenvectors) is given here.

SECTION 3.1. Introductory Examples, page 152

Purpose. In this section the student is supposed to gain a first impression of the importance of systems of differential equations in physics and engineering and why they occur, and why they lead to eigenvalue problems.

Main Content

Mixing problem

Electrical network

Conversion of single equations to systems [see (8)–(10)]

Background Material. Secs. 2.5, 2.11.

Short Courses. Take a quick look at Sec. 3.1, skip Sec. 3.2 and the beginning of Sec. 3.3, proceeding directly to solution methods in terms of the examples in Sec. 3.3.

Some Details on Content

Example 1 extends the physical system in Sec. 1.4, consisting of a single tank, to a system of two tanks. The principle of modeling remains the same. The problem leads to a typical eigenvalue problem, and the solutions show typical exponential increases and decreases.

Example 2 leads to a nonhomogeneous first-order system (a kind of system to be considered in Sec. 3.6). The vector **g** on the right in (5) causes a term + 3 in I_1, but has no effect on I_2, which is interesting to observe. If time permits, one could add a little discussion of particular solutions corresponding to different initial conditions.

Reduction of single equations to systems [formula (10)] is of great importance and should be emphasized. Example 3 illustrates it, and further applications follow in Sec. 3.5. It helps to create a "uniform" theory centered around first-order systems, along with the possibility of reducing higher order systems to first order.

SOLUTIONS TO PROBLEM SET 3.1, page 158

2. The system is

$$\begin{aligned} y_1' &= 0.02y_2 - 0.01y_1 \\ y_2' &= 0.01y_1 - 0.02y_2 \end{aligned}$$

where 0.01 appears because we divide by the content of the tank T_1, which is twice the old value. In proper order, the system becomes

$$\begin{aligned} y_1' &= -0.01y_1 + 0.02y_2 \\ y_2' &= 0.01y_1 - 0.02y_2. \end{aligned}$$

As a single vector equation,

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -0.01 & 0.02 \\ 0.01 & -0.02 \end{bmatrix}.$$

A has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -0.03$ and corresponding eigenvectors

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

respectively. The corresponding general solution is

$$\mathbf{y} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}e^{-0.03t}.$$

From the initial values,

$$\mathbf{y}(0) = c_1\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}.$$

In components this is $c_1 + c_2 = 0$, $0.5c_1 - c_2 = 150$. Hence $c_1 = 100$, $c_2 = -100$. This gives the solution

$$\mathbf{y} = 100\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} - 100\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-0.03t}.$$

In components,

$$\begin{aligned} y_1 &= 100(1 - e^{-0.03t}) \\ y_2 &= 100(\tfrac{1}{2} + e^{-0.03t}). \end{aligned}$$

4. In (6) we have

$$\mathbf{J}(t) = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ 0.8 \end{bmatrix}e^{-0.8t} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

From this and the initial conditions in vector form we get

$$\mathbf{J}(0) = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0.8\end{bmatrix} + \begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}28\\14\end{bmatrix};$$

in components,

$$\text{(a)} \quad 2c_1 + c_2 = 25$$

$$\text{(b)} \quad c_1 + 0.8c_2 = 14.$$

Subtract $\frac{1}{2}$(a) from (b) to get

$$0.3c_2 = 1.5, \qquad c_2 = 5.$$

Then from (a),

$$c_1 = \tfrac{1}{2}(25 - c_2) = 10.$$

Thus

$$\mathbf{J}(t) = \begin{bmatrix}20\\10\end{bmatrix}e^{-2t} + \begin{bmatrix}5\\4\end{bmatrix}e^{-0.8t} + \begin{bmatrix}3\\0\end{bmatrix};$$

in components,

$$I_1 = 20e^{-2t} + 5e^{-0.8t} + 3, \qquad I_2 = 10e^{-2t} + 4e^{-0.8t}.$$

6. The first differential equation remains as before. The second equation is obviously changed to

$$I_2' = 0.4I_1' - 0.54I_2.$$

Substitution of the first equation into the new second one, as in the text, gives

$$I_2' = -1.6I_1 + 1.06I_2 + 4.8.$$

Hence the matrix of the new system is

$$\mathbf{A} = \begin{bmatrix}-4 & 4\\-1.6 & 1.06\end{bmatrix}.$$

Its eigenvalues are -1.5 and -1.44. Corresponding eigenvectors are $\mathbf{x}^{(1)} = [1 \quad 0.625]^\mathsf{T}$ and $\mathbf{x}^{(2)} = [1 \quad 0.64]^\mathsf{T}$, respectively. The corresponding general solution is

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{-1.5t} + c_2\mathbf{x}^{(2)}e^{-1.44t}.$$

8. The system is

$$\begin{aligned} y_1' &= \qquad y_2 \\ y_2' &= -2y_1 - 3y_2. \end{aligned}$$

The matrix has the eigenvalues -1 and -2 and corresponding eigenvectors $[1 \quad -1]^\mathsf{T}$ and $[1 \quad -2]^\mathsf{T}$, respectively. From this

$$y = y_1 = c_1e^{-t} + c_2e^{-2t}$$

and the second equation gives the derivative $y_2 = y'$.

10. The system is

$$\begin{aligned} y_1' &= \qquad y_2 \\ y_2' &= y_1 + 15y_2/4. \end{aligned}$$

The matrix has the eigenvalues 4 and $-1/4$ and eigenvectors $[1 \quad 4]^T$ and $[1 \quad -\frac{1}{4}]^T$, respectively. The corresponding general solution is

$$y = c_1[1 \quad 4]^T e^{4t} + c_2[1 \quad -\tfrac{1}{4}]^T e^{-t/4}.$$

12. The system is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= 2y_1 + y_2 - 2y_3 \end{aligned}$$

The eigenvalues of its matrix are $1, -1, -2$. Eigenvectors are $[1 \quad 1 \quad 1]^T$, $[1 \quad -1 \quad 1]^T$, $[1 \quad -2 \quad 4]^T$, respectively. The corresponding general solution is

$$\mathbf{y} = c_1[1 \quad 1 \quad 1]^T e^t + c_2[1 \quad -1 \quad 1]^T e^{-t} + c_3[1 \quad -2 \quad 4]^T e^{-2t}.$$

14. TEAM PROJECT. (a) From Sec. 2.5 we know that the undamped motions of a mass on an elastic spring are governed by $my'' + ky = 0$ or

$$my'' = -ky$$

where $y = y(t)$ is the displacement of the mass. By the same arguments, for the two masses on the two springs in Fig. 77 we obtain the linear homogeneous system

$$\text{(11)} \qquad \begin{aligned} m_1y_1'' &= -k_1y_1 + k_2(y_2 - y_1) \\ m_2y_2'' &= -k_2(y_2 - y_1) \end{aligned}$$

for the unknown displacements $y_1 = y_1(t)$ of the first mass m_1 and $y_2 = y_2(t)$ of the second mass m_2. The forces acting on the first mass give the first equation, and the forces acting on the second mass give the second equation. Now $m_1 = m_2 = 1$, $k_1 = 3$, and $k_2 = 2$ in Fig. 77 so that by ordering (11) we obtain

$$\begin{aligned} y_1'' &= -5y_1 + 2y_2 \\ y_2'' &= 2y_1 - 2y_2 \end{aligned}$$

or, written as a single vector equation,

$$\mathbf{y}'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

(b) As for a single equation, we try an exponential function of t,

$$\mathbf{y} = \mathbf{x}e^{\omega t}. \qquad \text{Then} \qquad \mathbf{y}'' = \omega^2\mathbf{x}e^{\omega t} = \mathbf{A}\mathbf{x}e^{\omega t}.$$

Then, writing $\omega^2 = \lambda$ and dividing by $e^{\omega t}$, we get

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

Eigenvalues and eigenvectors are

$$\lambda_1 = -1, \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \lambda_2 = -6, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Since $\omega = \sqrt{\lambda}$ and $\sqrt{-1} = \pm i$ and $\sqrt{-6} = \pm i\sqrt{6}$, we get

$$\mathbf{y} = \mathbf{x}^{(1)}(c_1e^{it} + c_2e^{-it}) + \mathbf{x}^{(2)}(c_3e^{i\sqrt{6}t} + c_4e^{-i\sqrt{6}t})$$

or, by (7) in Sec. 2.3,

$$\mathbf{y} = a_1\mathbf{x}^{(1)}\cos t + b_1\mathbf{x}^{(1)}\sin t + a_2\mathbf{x}^{(2)}\cos\sqrt{6}t + b_2\mathbf{x}^{(2)}\sin\sqrt{6}t$$

where $a_1 = c_1 + c_2$, $b_1 = i(c_1 - c_2)$, $a_2 = c_3 + c_4$, $b_2 = i(c_3 - c_4)$. These four arbitrary constants can be specified by four initial conditions. In components, this solution is

$$\begin{aligned} y_1 &= a_1 \cos t + b_1 \sin t + 2a_2 \cos \sqrt{6}t + 2b_2 \sin \sqrt{6}t \\ y_2 &= 2a_1 \cos t + 2b_1 \sin t - a_2 \cos \sqrt{6}t - b_2 \sin \sqrt{6}t. \end{aligned}$$

(c) The conversion is done by the formulas

$$\begin{aligned} z_1 &= y_1, \quad z_1' = y_1' = z_2, \quad z_2' = y_1'' = -5z_1 + 2z_3 \\ z_3 &= y_2, \quad z_3' = y_2' = z_4, \quad z_4' = y_2'' = 2z_1 - 2z_3. \end{aligned}$$

This gives the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}.$$

Eigenvalues and eigenvectors are

$$-i, \begin{bmatrix} i \\ 1 \\ 2i \\ 2 \end{bmatrix}, \quad i, \begin{bmatrix} -i \\ 1 \\ -2i \\ 2 \end{bmatrix}, \quad -i\sqrt{6}, \begin{bmatrix} -2i\sqrt{6} \\ -12 \\ i\sqrt{6} \\ 6 \end{bmatrix}, \quad i\sqrt{6}, \begin{bmatrix} 2i\sqrt{6} \\ -12 \\ -i\sqrt{6} \\ 6 \end{bmatrix}.$$

Denoting these complex vectors by $\mathbf{z}^{(1)}, \cdots, \mathbf{z}^{(4)}$, we have as a general solution

$$\mathbf{z} = c_1\mathbf{z}^{(1)}e^{-it} + c_2\mathbf{z}^{(2)}e^{it} + c_3\mathbf{z}^{(3)}e^{-i\sqrt{6}\,t} + c_4\mathbf{z}^{(4)}e^{i\sqrt{6}\,t}.$$

The first and third components are

$$\begin{aligned} z_1 = y_1 &= ic_1e^{-it} - ic_2e^{it} - 2i\sqrt{6}c_3e^{-i\sqrt{6}\,t} + 2i\sqrt{6}c_4e^{i\sqrt{6}\,t} \\ z_3 = y_2 &= 2ic_1e^{-it} - 2ic_2e^{it} + i\sqrt{6}c_3e^{-i\sqrt{6}\,t} - i\sqrt{6}c_4e^{i\sqrt{6}\,t}. \end{aligned}$$

Converting this to real form by means of the Euler formula (Sec. 2.3) we obtain the same result as in (b), except for notations.

SECTION 3.2. Basic Concepts and Theory, page 159

Purpose. This survey of some basic concepts and facts on nonlinear and linear systems is intended to give the student an impression of the conceptual and structural similarity of the theory of systems to that of single differential equations.

Content, Important Concepts

Standard form of first-order systems

Form of corresponding initial value problems

Existence of solutions

Basis, general solution, Wronskian

Background Material. Sec. 2.7 contains the analogous theory for single equations. See also Sec. 1.9.

Short Courses. This section may be skipped, as mentioned before.

SECTION 3.3. Homogeneous Linear Systems with Constant Coefficients. Phase Plane, Critical Points, page 162

Purpose. Typical examples are intended to show the student the rich variety of pattern of solution curves (trajectories) near critical points in the phase plane, along with the process of actually solving homogeneous linear systems. This will also prepare the student for a good understanding of the systematic discussion of critical points in the phase plane in Sec. 3.4.

Main Content

Solution method for homogeneous linear systems

Examples illustrating types of critical points

Solution when no basis of eigenvectors is available

Important Concepts and Facts

Trajectories as solution curves in the phase plane

Phase plane as a means for the simultaneous (qualitative) discussion of a large number of solutions

Basis of solutions obtained from basis of eigenvectors

Background Material. Short review of eigenvalue problems from Sec. 3.0, if needed.

Short Courses. Omit Example 6.

Some Details on Content

In addition to developing skill in solving homogeneous linear systems, the student is supposed to become aware that it is the kind of eigenvalues that determine the type of critical point. The examples show important cases. (A systematic discussion of *all* cases follows in the next section.)

Example 1. Two negative eigenvalues give a **node.**

Example 2. A real double eigenvalue gives a **node.**

Example 3. Real eigenvalues of opposite sign give a **saddle point.**

Example 4. Pure imaginary eigenvalues give a **center,** and working in complex is avoided by a standard trick, which can also be useful in other contexts.

Example 5. Genuinely complex eigenvalues give a **spiral point.** Some work in complex can be avoided, if desired, by differentiation and elimination. The first equation is

$$\text{(a)} \qquad y_2 = y_1' + y_1.$$

By differentiation and from the second equation as well as (a),

$$y_1'' = -y_1' + y_2' = -y_1' - y_1 - (y_1' + y_1) = -2y_1' - 2y_1.$$

Complex solutions $e^{(-1\pm i)t}$ give the real solution

$$y_1 = e^{-t}(A \cos t + B \sin t).$$

From this and (a) follows the expression for y_2 given in the text.

Example 6 shows that the present method can be extended to include cases when **A** does not provide a basis of eigenvectors, but then becomes substantially more involved. In this way the student will recognize the importance of bases of eigenvectors, which also play a role in many other contexts.

SOLUTIONS TO PROBLEM SET 3.3, page 169

2. The eigenvalues are -3 and 4. Eigenvectors are $[2 \quad -5]^T$ and $[1 \quad 1]^T$, respectively. The corresponding general solution is

$$\begin{aligned} y_1 &= 2c_1e^{-3t} + c_2e^{4t} \\ y_2 &= -5c_1e^{-3t} + c_2e^{4t}. \end{aligned}$$

4. The eigenvalues are 3 and 9. Eigenvectors are $[3 \quad -1]^T$ and $[3 \quad 1]^T$, respectively. The corresponding general solution is

$$\begin{aligned} y_1 &= 3c_1e^{3t} + 3c_2e^{9t} \\ y_2 &= -c_1e^{3t} + c_2e^{9t}. \end{aligned}$$

6. The matrix has the double eigenvalue -6. An eigenvector is $[1 \quad -1]^T$. Hence the vector $\mathbf{u}$ needed is obtained from

$$(\mathbf{A} + 6\mathbf{I})\mathbf{u} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can take $\mathbf{u} = [0, \quad -\frac{1}{2}]$. With this we obtain as a general solution

$$\begin{aligned} y_1 &= c_1e^{-6t} + c_2te^{-6t} \\ y_2 &= -c_1e^{-6t} - c_2(t + \tfrac{1}{2})e^{-6t}. \end{aligned}$$

8. The eigenvalue -3 has two linearly independent eigenvectors, which we can choose as $[1 \quad 0 \quad 0]^T$ and $[0 \quad 2 \quad 1]^T$. The second eigenvalue is -6. A corresponding eigenvector is $[1 \quad 1 \quad -1]^T$. This gives the solution

$$\begin{aligned} y_1 &= c_1e^{-3t} + c_3e^{-6t} \\ y_2 &= 2c_2e^{-3t} + c_3e^{-6t} \\ y_3 &= c_2e^{-3t} - c_3e^{-6t}. \end{aligned}$$

10. $y_1 = 2e^{-3t} - 2e^{4t}, \quad y_2 = -5e^{-3t} - 2e^{4t}$

12. $y_1 = 3e^{3t} + e^{-t}, \quad y_2 = 6e^{3t} - 2e^{-t}$

14. $y_1 = -3e^{t} + 3e^{3t}, \quad y_2 = e^{t} + e^{3t}$

16. The restriction of the inflow from outside to pure water is necessary to obtain a homogeneous system. The principle involved in setting up the model is

$$\text{Time rate of change} = \text{Inflow} - \text{Outflow}.$$

For Tank T_1 this is (see Fig. 84)

$$y_1' = \left(12 \cdot 0 + \frac{4}{200}y_2\right) - \frac{16}{200}y_1.$$

For Tank T_2 it is

$$y_2' = \frac{16}{200}y_1 - \frac{4 + 12}{200}y_2.$$

Performing the divisions and ordering terms, we have

$$\begin{aligned} y_1' &= -0.08y_1 + 0.02y_2 \\ y_2' &= 0.08y_1 - 0.08y_2. \end{aligned}$$

The eigenvalues of the matrix of this system are -0.04 and -0.12. Eigenvectors are

$[1 \quad 2]^T$ and $[1 \quad -2]^T$, respectively. The corresponding general solution is

$$\mathbf{y} = c_1\begin{bmatrix}1\\2\end{bmatrix}e^{-0.04t} + c_2\begin{bmatrix}1\\-2\end{bmatrix}e^{-0.12t}.$$

The initial condition is $y_1(0) = 100$, $y_2(0) = 200$. This gives $c_1 = 100$, $c_2 = 0$. In components the answer is

$$y_1 = 100e^{-0.04t}$$
$$y_2 = 200e^{-0.04t}.$$

Both functions approach zero as $t \to \infty$, a reasonable result because pure water flows in and mixture flows out.

18. Differentiate the first given equation,

$$I_1/C + R(I_1' - I_2') = 0.$$

Solve algebraically for I_1', substituting I_2' from the second given equation. Solve the second given equation algebraically for I_2'. Then we have the system in the usual form

$$I_1' = \left(\frac{R}{L} - \frac{1}{RC}\right)I_1 - \frac{R}{L}I_2$$
$$I_2' = \frac{R}{L}I_1 - \frac{R}{L}I_2.$$

Thus the matrix is

$$\mathbf{A} = \begin{bmatrix} R/L - 1/RC & -R/L \\ R/L & -R/L \end{bmatrix}.$$

This gives the characteristic equation

$$\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC} = 0$$

and the eigenvalues

$$\lambda = -\frac{1}{2RC} \pm \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}}.$$

Hence the eigenvalues are real if and only if

$$\frac{1}{4R^2C^2} \geqq \frac{1}{LC}, \qquad \text{thus,} \qquad L \geqq 4R^2C.$$

20. TEAM PROJECT. From the complex solution in Example 4 we can obtain a real basis and a real general solution by the Euler formula (Sec. 2.3), which we need in the form

$$e^{2it} = \cos 2t + i \sin 2t, \qquad e^{-2it} = \cos 2t - i \sin 2t.$$

Collecting the real and imaginary parts, we thus obtain in the complex solution (12*)

$$\text{(A)} \quad \begin{bmatrix}1\\2i\end{bmatrix}e^{2it} = \begin{bmatrix}1\\2i\end{bmatrix}(\cos 2t + i \sin 2t) = \begin{bmatrix}\cos 2t\\-2\sin 2t\end{bmatrix} + i\begin{bmatrix}\sin 2t\\2\cos 2t\end{bmatrix}$$

and similarly

$$\begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it} = \begin{bmatrix} 1 \\ -2i \end{bmatrix} (\cos 2t - i \sin 2t) = \begin{bmatrix} \cos 2t \\ -2 \sin 2t \end{bmatrix} - i \begin{bmatrix} \sin 2t \\ 2 \cos 2t \end{bmatrix}.$$

Substitution into (12) shows that the real part and the imaginary part in (A),

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos 2t \\ -2 \sin 2t \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sin 2t \\ 2 \cos 2t \end{bmatrix},$$

are solutions. These real solutions form a basis because their Wronskian is not zero,

$$W(\mathbf{u}, \mathbf{v}) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2 \sin 2t & 2 \cos 2t \end{vmatrix} = 2 \cos^2 2t + 2 \sin^2 2t = 2.$$

Hence a real general solution of (12) is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\mathbf{u} + B\mathbf{v} = A \begin{bmatrix} \cos 2t \\ -2 \sin 2t \end{bmatrix} + B \begin{bmatrix} \sin 2t \\ 2 \cos 2t \end{bmatrix}.$$

This represents the same ellipses as before because by calculation and simplification we find

$$y_1^2 + \tfrac{1}{4} y_2^2 = (A^2 + B^2)(\cos^2 2t + \sin^2 2t) = A^2 + B^2 = const.$$

We turn to ***Example (5).*** The complex solution is

$$\text{(B)} \qquad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}.$$

We derive from this a *real* general solution. In (B) we have

$$\begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} = \begin{bmatrix} e^{-t}(\cos t + i \sin t) \\ ie^{-t}(\cos t + i \sin t) \end{bmatrix} = \begin{bmatrix} e^{-t} \cos t \\ -e^{-t} \sin t \end{bmatrix} + i \begin{bmatrix} e^{-t} \sin t \\ e^{-t} \cos t \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t} = \begin{bmatrix} e^{-t}(\cos t - i \sin t) \\ -ie^{-t}(\cos t - i \sin t) \end{bmatrix} = \begin{bmatrix} e^{-t} \cos t \\ -e^{-t} \sin t \end{bmatrix} - i \begin{bmatrix} e^{-t} \sin t \\ e^{-t} \cos t \end{bmatrix}.$$

The real and imaginary parts on the right are real solutions of (13)—call them **u** and **v**—as can be seen by substitution. They form a basis because their Wronskian is not zero.

$$W(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} e^{-t} \cos t & e^{-t} \sin t \\ -e^{-t} \sin t & e^{-t} \cos t \end{bmatrix} = e^{-2t}(\cos^2 t + \sin^2 t) = e^{-2t}.$$

The corresponding real general solution is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\mathbf{u} + B\mathbf{v} = A \begin{bmatrix} e^{-t} \cos t \\ -e^{-t} \sin t \end{bmatrix} + B \begin{bmatrix} e^{-t} \sin t \\ e^{-t} \cos t \end{bmatrix};$$

in components,

$$\text{(C)} \qquad y_1 = e^{-t}(A \cos t + B \sin t), \qquad y_2 = e^{-t}(B \cos t - A \sin t).$$

It represents spirals (see Fig. 82). To see this, we introduce the usual polar coordinates r, θ in the y_1y_2-plane defined by

$$r^2 = y_1^2 + y_2^2, \qquad \tan \theta = y_2/y_1.$$

Then by straightforward calculation and simplification of the result we obtain from (C)

$$r^2 = (A^2 + B^2)e^{-2t}, \qquad \text{thus} \qquad r = c_0 e^{\theta},$$

where $c_0 = \sqrt{A^2 + B^2}$ and $\theta = -t$. For each c_0 this represents a spiral, as claimed. The origin is a spiral point of the system (13).

SECTION 3.4. Criteria for Critical Points. Stability, page 170

Purpose. Systematic discussion of critical points in the phase plane from the standpoints of both the geometrical shapes of trajectories and **stability.**

Main Content

Formula (9) for the types of critical points

Formula (10) for the stability behavior

Stability chart, giving (9) and (10) graphically

Important Concepts

Node, saddle point, center, spiral point

Stable and attractive, stable, unstable

Background Material. Sec. 2.5 (needed in Example 2).

Short Courses. Since all those types of critical points already occurred in the previous section, one may perhaps present just a short discussion of stability.

Some Details on Content

The types of critical points in Sec. 3.3 now recur, and the discussion shows that they exhaust all possibilities. With the examples of Sec. 3.3 fresh in mind, the student will acquire a deeper understanding by discussing the **stability chart** and by reconsidering those examples from the viewpoint of stability. This gives the instructor an opportunity to emphasize that the general importance of stability in engineering can hardly be overestimated.

Example 2, relating to the familiar free vibrations in Sec. 2.5, gives a good illustration of stability behavior, namely, depending on c, attractive stability, stability (and instability if one includes "negative damping," with $c < 0$, as it will recur in the next section in connection with the famous van der Pol equation).

SOLUTIONS TO PROBLEM SET 3.4, page 174

2. $p = 0$, $q = -9$, saddle point, always unstable. A general solution is

$$\begin{aligned} y_1 &= c_1 e^{-3t} + c_2 e^{3t} \\ y_2 &= -5c_1 e^{-3t} + c_2 e^{3t}. \end{aligned}$$

4. $p = -12$, $q = 27$, $\Delta = 144 - 108 > 0$, stable and attractive node. A general solution is

$$\begin{aligned} y_1 &= c_1 e^{-3t} + c_2 e^{-9t} \\ y_2 &= -3c_1 e^{-3t} + 3c_2 e^{-9t}. \end{aligned}$$

6. $p = 0$, $q = 9$, $\Delta = -36$, center, always stable. A complex general solution, as obtained directly from the characteristic equation, is

$$\mathbf{y} = c_1\begin{bmatrix} 1 - 3i \\ 5 \end{bmatrix} e^{-3it} + c_2\begin{bmatrix} 1 + 3i \\ 5 \end{bmatrix} e^{3it}.$$

The conversion to real form takes patience:

1. Take the simpler of the two components and multiply everything out. Then collect the cosine and the sine terms and choose a notation for their coefficients, say, A and B.
2. Express c_1 and c_2 in terms of A and B.
3. Substitute the result just obtained into the first component and simplify.

In the present case the second component, y_2, is simpler:

$$\begin{aligned} y_2 &= 5c_1(\cos 3t - i \sin 3t) + 5c_2(\cos 3t + i \sin 3t) \\ &= (5c_1 + 5c_2)\cos 3t + (-5ic_1 + 5ic_2)\sin 3t \\ &= 10A\cos 3t + 10B\sin 3t \end{aligned}$$

where

$$\begin{aligned} 5c_1 + 5c_2 &= 10A \\ 5c_1 - 5c_2 &= 10iB. \end{aligned}$$

In the second step we solve this for c_1 and c_2, obtaining

$$\begin{aligned} c_1 &= A + iB \\ c_2 &= A - iB. \end{aligned}$$

In the third step we turn to the first component,

$$\begin{aligned} y_1 &= c_1(1 - 3i)(\cos 3t - i \sin 3t) + c_2(1 + 3i)(\cos 3t + i \sin 3t) \\ &= [(1 - 3i)c_1 + (1 + 3i)c_2]\cos 3t \\ &\quad + [(-i(1 - 3i)c_1 + i(1 + 3i)c_2]\sin 3t. \end{aligned}$$

Expressing c_1 and c_2 in terms of A and B and simplifying (in this operation, imaginary terms must drop out by cancellation) we obtain

$$y_1 = (2A + 6B)\cos 3t + (2B - 6A)\sin 3t.$$

8. $p = -3$, $q = -10$, saddle point, always unstable. A general solution is

$$\begin{aligned} y_1 &= c_1e^{-5t} + 4c_2e^{2t} \\ y_2 &= -c_1e^{-5t} + 3c_2e^{2t}. \end{aligned}$$

10. We could solve the first equation, $y_1 = c_1e^{-t}$ and insert this into the second equation, and solve it. Or we can follow the rule. An eigenvalue is -1 and has only a single independent eigenvector, say, $[0 \quad 1]^T$, so that we have no eigenbasis and have to determine $\mathbf{u}$ from

$$(\mathbf{A} + \mathbf{I})\mathbf{u} = \begin{bmatrix} 0 & 0 \\ -5 & 0 \end{bmatrix}\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This gives $-5u_1 = 1$, $u_1 = -1/5$, $u_2 = 0$. A general solution is

$$y_1 = \qquad -\tfrac{1}{5}c_2e^{-t}$$
$$y_2 = c_1e^{-t} + c_2te^{-t}.$$

The critical point is a degenerate node, which is stable and attractive.

12. $y = A\cos\frac{1}{3}t + B\sin\frac{1}{3}t$. The trajectories are the ellipses

$$\tfrac{1}{9}{y_1}^2 + {y_2}^2 = const.$$

This is obtained as in Example 4 in Sec. 3.3.

14. $y = e^{-t}(A\cos t + B\sin t)$. The trajectories are stable and attractive spirals.

16. $y_1' = -dy_1/d\tau$, $y_2' = -dy_2/d\tau$, reversal of the direction of motion; to get the usual form, we have to multiply the transformed system by -1, which amounts to multiplying the matrix by -1, changing p into $-p$, but leaving q and Δ unchanged. In the example, we get an unstable node.

18. At a center, $p = a_{11} + a_{22} = 0$, $q = \det \mathbf{A} > 0$, hence $\Delta < 0$. Under the change, p changes into $a_{11} + k + a_{22} + k = 2k \neq 0$; q remains positive because

$$(a_{11} + k)(a_{22} + k) - a_{12}a_{21} = q + k^2 > 0.$$

Finally, Δ remains unchanged because

$$(p + 2k)^2 - 4(q + k^2) = (2k)^2 - 4(q + k^2) = -4q < 0.$$

Hence we obtain a spiral point, which is unstable if $k > 0$ and stable and attractive if $k < 0$.

We can reason more simply as follows. For a center the eigenvalues are pure imaginary (to have closed trajectories). An eigenvalue λ of $\mathbf{A}$ gives an eigenvalue $\lambda + k$ of $\mathbf{A}$, causing a damped oscillation (when $k < 0$) or an increasing one (when $k > 0$), thus a spiral.

SECTION 3.5. Qualitative Methods for Nonlinear Systems, page 175

Purpose. As a most important step, in this section we extend phase plane methods to nonlinear systems and nonlinear equations.

Main Content

Critical points of nonlinear systems

Their discussion by linearization

Transformation of single autonomous equations

Applications of linearization and transformation techniques

Important Concepts and Facts

Linearized system (3), condition for applicability

Linearization of pendulum equations

Self-sustained oscillations, van der Pol equation

Short Courses. Linearization at different critical points seems the main issue that the student is supposed to understand and handle practically. Examples 1 and 2 may help students gain skill in that technique. The other material could be skipped without loss of continuity.

Some Details on Content

This section is very important, because from it the student should learn not only techniques (linearization, etc.) but also the fact that phase plane methods are particularly powerful and important in application to systems or single equations that cannot be solved explicitly. The student should also recognize that it is quite surprising how much information these methods can give. This is demonstrated by the **pendulum equation** (Examples 1 and 2) for a relatively simple system, and by the famous **van der Pol equation** for a single equation, which has become a prototype for self-sustained oscillations of electrical systems of various kinds.

We also discuss the famous **Lotka–Volterra predator–prey model.**

For the **Rayleigh** and **Duffing equations,** see the problem set.

SOLUTIONS TO PROBLEM SET 3.5, page 183

2. $y = A\cos t + B\sin t$, radius $\sqrt{A^2 + B^2}$

4. $(n\pi, 0)$ saddle points for even n and centers for odd n

6. At $(0, 0)$, $y_1' = y_2$, $y_2' = -y_1$, $p = 0$, $q = 1$, $\Delta = -4$, center. The other critical point is at $(-1, 0)$. We set $y_1 = -1 + \tilde{y}_1$, $y_2 = \tilde{y}_2$. Then $-y_1 - y_1^2 \approx \tilde{y}_1$. Hence $\tilde{y}_1' = \tilde{y}_2$, $\tilde{y}_2' = \tilde{y}_1$. This gives a saddle point.

8. $y'' + y - y^3 = 0$ written as a system is

$$y_1' = y_2$$
$$y_2' = -y_1 + y_1^3.$$

Now $-y_1 + y_1^3 = y_1(-1 + y_1^2) = 0$ shows that there are three critical points, at $(y_1, y_2) = (0, 0)$, $(-1, 0)$, and $(1, 0)$.

The linearized system at $(0, 0)$ is

$$y_1' = y_2 \qquad \text{Matrix:} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$
$$y_2' = -y_1.$$

From the matrix we see that $p = a_{11} + a_{22} = 0$, $q = 1$. Hence $(0, 0)$ is a center (see Sec. 3.4).

For the next critical point we have to linearize at $(-1, 0)$ by setting

$$y_1 = -1 + \tilde{y}_1, \qquad y_2 = \tilde{y}_2.$$

Then

$$y_1(-1 + y_1^2) = (-1 + \tilde{y}_1)[-1 + (-1 + \tilde{y}_1)^2]$$
$$= (-1 + \tilde{y}_1)[-2\tilde{y}_1 + \tilde{y}_1^2] \approx 2\tilde{y}_1.$$

Hence the linearized system is

$$\tilde{y}_1' = \tilde{y}_2 \qquad \text{Matrix:} \quad \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$
$$\tilde{y}_2' = 2\tilde{y}_1.$$

Hence $q = \det \mathbf{A} = -2 < 0$, that is, the critical point at $(-1, 0)$ is a saddle point.

Similarly, to linearize at $(1, 0)$, set

$$y_1 = 1 + \tilde{y}_1, \qquad y_2 = \tilde{y}_2.$$

Then

$$-y_1 + y_1^3 \approx 2\tilde{y}_1$$

and we obtain another saddle point, as just before.

10. The equation gives the system

$$y_1' = y_2$$
$$y_2' = -4y_1 + 5y_1^3 - y_1^5.$$

Now

$$f(y_1) = -4y_1 + 5y_1^3 - y_1^5 = -y_1(4 - 5y_1^2 + y_1^4) = 0$$

involves a quadratic equation in y_1^2 with solutions $y_1^2 = 1, 4$. Hence the zeros of $f(y_1)$ are $\pm 2, \pm 1, 0$ and give the five critical points $(y_1, 0)$ with $y_1 = -2, -1, 0, 1, 2$.

Linearization leads to the result that (0, 0), (−2, 0), and (2, 0) are centers and (−1, 0), (1, 0) are saddle points.

For instance, at (0, 0), linearize to

$$y_1' = y_2, \quad y_2' = -4y_1 \qquad \text{to get} \qquad p = 0, \quad q = 4 > 0, \quad \text{a center.}$$

At the other points some work may be saved by setting

$$f(y_1) = -y_1(y_1 + 1)(y_1 - 1)(y_1 + 2)(y_1 - 2)$$

and substituting $y_1 = -2 + \tilde{y}_1$ at (−2, 0) (etc. for the others) and finding

$$f(y_1) = -(\tilde{y}_1 - 2)(\tilde{y}_1 - 1)(\tilde{y}_1 - 3)(\tilde{y}_1)(\tilde{y}_1 - 4) \approx -24\tilde{y}_1,$$

giving the linearized system

$$\tilde{y}_1' = \tilde{y}_2, \quad \tilde{y}_2' = -24\tilde{y}_1, \qquad p = 0, \quad q = 24, \quad \text{a center.}$$

More savings follow by noting that every linearized system is of the form

$$\tilde{\mathbf{y}}' = \begin{bmatrix} 0 & 1 \\ \tilde{a}_{21} & 0 \end{bmatrix} \tilde{\mathbf{y}}, \qquad \text{thus} \qquad p = 0, \quad q = -\tilde{a}_{21}.$$

Now

$$q = -\tilde{a}_{21} = -f'(y_1) = 4 - 15y_1^2 + 5y_1^4$$

is positive at 0 and ± 2, thus giving centers, and negative at ± 1, giving saddle points, as asserted.

12. $y_1' = y, y_2' = 4y_1 - y_1^3, y_2 y_2' = 4y_1 y_1' - y_1^3 y_1',$
$y_2^2 = 4y_1^2 - \frac{1}{2}y_1^4 + c^*$ or (see the figure on the next page)

$$y_2^2 = \tfrac{1}{2}(c + 4 - y_1^2)(c - 4 + y_1^2).$$

14. Critical points at (0, 0), (2, 0), (−2, 0). Linearization leads to the following:

At (0, 0): $\quad y_1' = y_2, \quad y_2' = 4y_1 - y_2, \qquad q = -4, \quad$ saddle point.

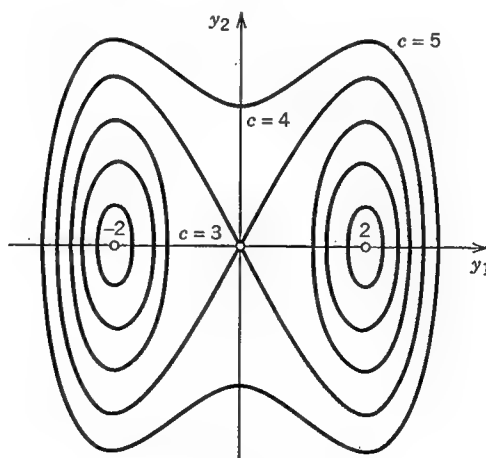
At (±2, 0): $\quad y_1' = y_2, \quad y_2' = -8y_1 - y_2, \qquad p = -1, q = 8, \Delta = 1 - 32 < 0,$

which gives stable and attractive spiral points (instead of centers).

Note the similarity to the situation in the case of the undamped and damped pendulum equations.

16. TEAM PROJECT. (a) Unstable node if $\mu \geqq 2$, unstable spiral point if $2 > \mu > 0$, center if $\mu = 0$, stable and attractive spiral point if $0 > \mu > -2$, stable and attractive node if $\mu \leqq -2$.

(c) $y_1' = y_2, y_2' = -(\omega_0^2 y_1 + \beta y_1^3)$, hence

$$y_2' y_2 = -(\omega_0^2 y_1 + \beta y_1^3) y_1'.$$

Section 3.5. Problem 12

By integration on both sides,

$$y_2^2 + \omega_0^2 y_1^2 + \tfrac{1}{2}\beta y_1^4 = const.$$

SECTION 3.6. Nonhomogeneous Linear Systems, page 184

Purpose. We now turn from homogeneous linear systems considered so far to solution methods for nonhomogeneous systems.

Main Content

Method of undetermined coefficients

Modification for special right sides

Method of variation of parameters

Method of diagonalization

Short Courses. Select just one or two of the preceding methods.

Some Details on Content

In addition to understanding the solution methods as such, the student should observe the conceptual and technical similarities to the handling of nonhomogeneous linear differential equations in Secs. 2.8–2.12 and 2.15 and understand the reason for this, namely, that systems can be converted to single equations and conversely. For instance, in connection with Example 2 in this section, one may point to the Modification Rule in Sec. 2.9, or, if time permits, establish an even more definite relation by differentiation and elimination of y_2,

$$\begin{aligned} y_1'' &= -3y_1' + y_2' + 12e^{-2t} \\ &= -3y_1' + (y_1 - 3y_2 + 2e^{-2t}) + 12e^{-2t} \\ &= -3y_1' + y_1 - 3(y_1' + 3y_1 + 6e^{-2t}) + 14e^{-2t} \\ &= -6y_1' - 8y_1 - 4e^{-2t}, \end{aligned}$$

solving this for y_1 and then getting y_2 from the solution.

SOLUTIONS TO PROBLEM SET 3.6, page 189

2. The eigenvalues are -2 and 2. Eigenvectors are $[1 \quad -1]^T$ and $[1 \quad 1]^T$, respectively. A particular solution can be obtained by the method of undetermined coefficients. *Answer:*

$$y_1 = c_1e^{-2t} + c_2e^{2t} - \tfrac{3}{4}, \qquad y_2 = -c_1e^{-2t} + c_2e^{2t} - \tfrac{1}{2}t.$$

4. The eigenvalues are -1 and 2, with eigenvectors $[1 \quad 1]^T$ and $[1 \quad \tfrac{1}{4}]^T$, respectively. By the method of undetermined coefficients we have to assume, say, $y_1 = A_1 \cos t + B_1 \sin t$; similarly for y_2. *Answer:*

$$y_1 = c_1e^{-t} + c_2e^{2t} - 7\cos t + \sin t$$

$$y_2 = c_1e^{-t} + \tfrac{1}{4}c_2e^{2t} - 3\cos t - \sin t.$$

6. The eigenvalues are 2 and 5, with eigenvectors $[1 \quad -2]^T$ and $[1 \quad 1]^T$, respectively. *Answer:*

$$y_1 = c_1e^{2t} + c_2e^{5t} - 0.18 - 0.4t$$

$$y_2 = -2c_1e^{2t} + c_2e^{5t} + 0.32 + 0.6t.$$

8. From the characteristic equation we obtain

$$\lambda_1 = 4, \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -4, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

By the method of undetermined coefficients, we set

$$\mathbf{y}^{(p)} = \mathbf{u} + \mathbf{v}t + \mathbf{w}t^2.$$

By substitution,

$$\mathbf{y}^{(p)\prime} = \mathbf{v} + 2\mathbf{w}t = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}(\mathbf{u} + \mathbf{v}t + \mathbf{w}t^2) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -16 \end{bmatrix}t^2.$$

We now compare componentwise the constant terms, linear terms, and quadratic terms:

$$\left.\begin{aligned} v_1 &= 4u_2 \\ v_2 &= 4u_1 + 2 \end{aligned}\right\} \text{constant terms}$$

$$\left.\begin{aligned} 2w_1 &= 4v_2 \\ 2w_2 &= 4v_1 \end{aligned}\right\} \text{linear terms}$$

$$\left.\begin{aligned} 0 &= 4w_2 \\ 0 &= 4w_1 - 16 \end{aligned}\right\} \text{quadratic terms.}$$

We then obtain, in this order,

$$w_1 = 4, \quad w_2 = 0, \quad v_1 = 0, \quad v_2 = 2, \quad u_1 = \tfrac{1}{4}(v_2 - 2) = 0, \quad u_2 = 0.$$

The corresponding general solution is

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{4t} + c_2\mathbf{x}^{(2)}e^{-4t} + \mathbf{v}t + \mathbf{w}t^2;$$

in components,

$$y_1 = c_1e^{4t} + c_2e^{-4t} + 4t^2, \qquad y_2 = c_1e^{4t} - c_2e^{-4t} + 2t.$$

From this and the initial conditions $y_1(0) = 3$, $y_2(0) = 1$ we obtain $c_1 = 2$, $c_2 = 1$.

Answer:

$$y_1 = 2e^{4t} + e^{-4t} + 4t^2, \qquad y_2 = 2e^{4t} - e^{-4t} + 2t.$$

10. From the characteristic equation,

$$\lambda_1 = 2, \quad \mathbf{x}^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -4, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\mathbf{y}^{(p)}$ can be obtained by the method of undetermined coefficients, starting from

$$\mathbf{y}^{(p)} = \mathbf{a} \cosh t + \mathbf{b} \sinh t.$$

Substitution gives

$$\begin{aligned} \mathbf{y}^{(p)\prime} &= \mathbf{a} \sinh t + \mathbf{b} \cosh t \\ &= \begin{bmatrix} 4 & -8 \\ 2 & -6 \end{bmatrix} (\mathbf{a} \cosh t + \mathbf{b} \sinh t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cosh t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sinh t. \end{aligned}$$

Comparing sinh terms and cosh terms (componentwise), we get from this

$$\left.\begin{aligned} a_1 &= 4b_1 - 8b_2 \\ a_2 &= 2b_1 - 6b_2 + 2 \end{aligned}\right\} \text{sinh terms}$$

$$\left.\begin{aligned} b_1 &= 4a_1 - 8a_2 + 2 \\ b_2 &= 2a_1 - 6a_2 + 1 \end{aligned}\right\} \text{cosh terms.}$$

To solve this, one can substitute the first two equations into the last two, solve for $b_1 = 2$, $b_2 = 1$, and then get from the first two equations $a_1 = a_2 = 0$. This gives the general solution

$$y_1 = 4c_1e^{2t} + c_2e^{-4t} + 2 \sinh t, \quad y_2 = c_1e^{2t} + c_2e^{-4t} + \sinh t.$$

From the initial conditions we see that $c_1 = 0$, $c_2 = 0$, so that the general solution does not contribute to the answer. This is not automatically the case when we have $y_1(0) = 0$, $y_2(0) = 0$, but is a consequence of the fact that $\mathbf{y}^{(p)}$ at $t = 0$ is the zero vector. *Answer:* $y_1 = 2 \sinh t$, $y_2 = \sinh t$.

12. $\mathbf{y} = c_1\mathbf{x}^{(1)}e^t + c_2\mathbf{x}^{(2)}e^{2t}$, $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{x}^{(2)} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$. Now e^t on the right is a solution of the homogeneous system. Hence, to find $\mathbf{y}^{(p)}$, we have to proceed as in Example 2, setting

$$\mathbf{y}^{(p)} = \mathbf{u}te^t + \mathbf{v}e^t.$$

Substitution gives

$$\mathbf{y}^{(p)\prime} = \mathbf{u}(t + 1)e^t + \mathbf{v}e^t = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} (\mathbf{u}te^t + \mathbf{v}e^t) + \begin{bmatrix} 5 \\ -6 \end{bmatrix} e^t.$$

Equating the terms in e^t (componentwise) gives

$$\begin{aligned} u_1 + v_1 &= -3v_1 - 4v_2 + 5 \\ u_2 + v_2 &= 5v_1 + 6v_2 - 6 \end{aligned}$$

and the terms in te^t give

$$\begin{aligned} u_1 &= -3u_1 - 4u_2 \\ u_2 &= 5u_1 + 6u_2. \end{aligned}$$

Hence $u_1 = 1$, $u_2 = -1$, $v_1 = 1$, $v_2 = 0$. This gives the general solution

$$y_1 = c_1e^t + 4c_2e^{2t} + te^t + e^t, \qquad y_2 = -c_1e^t - 5c_2e^{2t} - te^t.$$

From the initial conditions we obtain $c_1 = -2$, $c_2 = 5$. *Answer:*

$$y_1 = -2e^t + 20e^{2t} + te^t + e^t, \qquad y_2 = 2e^t - 25e^{2t} - te^t.$$

14. A general solution of the homogeneous system is

$$\mathbf{y}^{(h)} = c_1\mathbf{x}^{(1)}e^{3t} + c_2\mathbf{x}^{(2)}e^{-t}, \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Answer:

$$y_1 = 2e^{-t} + t^2, \qquad y_2 = -e^{-t} - t.$$

16. The formula for $\mathbf{v}$ shows that these various choices differ by multiples of the eigenvector for $\lambda = -2$, which can be absorbed into, or taken out of, c_1 in the general solution $\mathbf{y}^{(h)}$.

18. The equations are

(a) $$I_1' = -2I_1 + 2I_2 + 440 \sin t$$

and

$$8I_2 + 2\int I_2\,dt + 2(I_2 - I_1) = 0.$$

Thus

$$I_2 = -0.25\int I_2\,dt + 0.25(I_1 - I_2),$$

which, upon differentiation and insertion of I_1' from (a) and simplification, gives

(b) $$I_2' = -0.4I_1 + 0.2I_2 + 88 \sin t.$$

The general solution of the homogeneous system is as in Prob. 17, and the method of undetermined coefficients gives as a particular solution

$$-\frac{1}{3}\begin{bmatrix} 352 \\ 44 \end{bmatrix}\cos t + \frac{1}{3}\begin{bmatrix} 616 \\ 132 \end{bmatrix}\sin t.$$

20. $$\mathbf{A} = \begin{bmatrix} -3 & 1.25 \\ 1 & -1 \end{bmatrix},$$

$$\mathbf{J} = -\frac{125}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-0.5t} - \frac{125}{21}\begin{bmatrix} 5 \\ -2 \end{bmatrix}e^{-3.5t} + \frac{500}{7}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

SOLUTIONS TO CHAPTER 3 REVIEW, page 190

16. Eigenvalues -1, 6. Eigenvectors $[1 \quad -1]^{\mathsf{T}}$, $[4 \quad 3]^{\mathsf{T}}$. The corresponding general solution is

$$y_1 = c_1e^{-t} + 4c_2e^{6t}, \qquad y_2 = -c_1e^{-t} + 3c_2e^{6t}.$$

The critical point at $(0, 0)$ is a saddle point, which is always unstable.

18. Eigenvalues $\pm 2i$. Hence $y_1 = A\cos 2t + B\sin 2t$. From this and the first equation,

$$y_2 = -\tfrac{1}{2}y_1' = -B\cos 2t + A\sin 2t.$$

The critical point at (0, 0) is a center, which is always stable.

20. $y_1 = c_1e^{-t} + c_2e^{-2t}$, $y_2 = -2c_1e^{-t} - 3c_2e^{-2t}$, stable and attractive node

22. $y_1 = c_1e^{3t} + c_2e^{-t}$, $y_2 = \frac{1}{2}c_1e^{3t} - \frac{1}{2}c_2e^{-t}$; saddle point

24. $y_1 = 2c_1e^t + c_2e^{-2t} + 2te^t + e^t$
$y_2 = -c_1e^t - 2c_2e^{-2t} - te^t$

26. $y_1 = 2c_1e^{-t} + 2c_2e^{3t} + \cos t - \sin t$, $y_2 = -c_1e^{-t} + c_2e^{3t}$

28. $q = \lambda_1\lambda_2 < 0$ for a saddle point, so that λ_1 and λ_2 must be real. $\mathbf{A}^2$ has the eigenvalues $\mu_1 = \lambda_1{}^2$ and $\mu_2 = \lambda_2{}^2$, which are real, and $\mu_1\mu_2 > 0$, as well as

$$(\mu_1 + \mu_2)^2 - 4\mu_1\mu_2 = (\mu_1 - \mu_2)^2 \geqq 0,$$

so that we get a node, which is unstable because $\mu_1 + \mu_2 > 0$.

30. The matrix of the system is

$$\begin{bmatrix} A - B & -A \\ A & -A \end{bmatrix},$$

where $A = R/L$ and $B = 1/RC$. A general solution is

$$\mathbf{y} = c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 4 \\ -1 \end{bmatrix}e^{-4t}$$

and the initial conditions give $c_1 = -1/3$ and $c_2 = 1/3$.

32. $I_1 - I_2 + 10I_1' = 100$, $I_2 + I_2' - I_1' = 0$ (after differentiation). Solve the first equation algebraically for I_1'. Replace I_1' in the second equation by using the first equation. This gives the system

$$I_1' = -0.1I_1 + 0.1I_2 + 10$$
$$I_2' = -0.1I_1 - 0.9I_2 + 10.$$

Eigenvalues $\lambda = -0.5 \pm \sqrt{0.15} = -0.1127, -0.8873$; corresponding eigenvectors:

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ -0.127 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -7.873 \end{bmatrix}.$$

Answer:

$$\mathbf{J} = c_1\mathbf{x}^{(1)}e^{-0.1127t} + c_2\mathbf{x}^{(2)}e^{-0.8873t} + \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

with $c_1 = -101.6$ and $c_2 = 1.64$ from the initial conditions.

34. A general solution is

$$y_1 = A\cos t + B\sin t + 1$$
$$y_2 = B\cos t - A\sin t - t.$$

(I) *Undetermined coefficients.* This is much simpler than the other two methods. $y_1 = At + B$, $y_2 = Ct + D$. By substitution, $A = Ct + D + t$, $C = -At - B$; thus $C = -1$, $A = 0$, $B = 1$, $D = 0$.

(II) *Variation of parameters.* We write $c = \cos t$, $s = \sin t$. Then

$$\mathbf{y}^{(p)} = \mathbf{Yu} = \mathbf{Y}\int_0^t \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} \tilde{t} \\ 0 \end{bmatrix} d\tilde{t} = \mathbf{Y}\int_0^t \begin{bmatrix} \tilde{t}\cos\tilde{t} \\ \tilde{t}\sin\tilde{t} \end{bmatrix} d\tilde{t}$$

$$= \begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{bmatrix} ts + c - 1 \\ -tc + s \end{bmatrix} = \begin{bmatrix} 1 - c \\ -t + s \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -t \end{bmatrix} - \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix},$$

where the last term is a solution of the homogeneous system.

(III) *Diagonalization* can be done in complex. $\lambda_1 = i$, $\lambda_2 = -i$, and

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{AX} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

as expected. Also,

$$\mathbf{h} = \mathbf{X}^{-1}\mathbf{g} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} t \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} t \\ t \end{bmatrix}.$$

Thus

$$z_1' = iz_1 + \tfrac{1}{2}t$$
$$z_2' = -iz_1 + \tfrac{1}{2}t$$

Particular solutions are

$$z_1 = e^{it}\int e^{-it}\tfrac{1}{2}t\,dt = \tfrac{1}{2}(it + 1)$$
$$z_2 = e^{-it}\int e^{it}\tfrac{1}{2}t\,dt = \tfrac{1}{2}(-it + 1)$$

and thus

$$\mathbf{y}^{(p)} = \mathbf{Xz} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} it/2 + 1/2 \\ -it/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -t \end{bmatrix}.$$

36. $(n\pi, 0)$ centers (n integer)

38. Critical points at $(0, 0)$ and $(0, -1)$. The linearized systems are

$$y_1' = 2y_2 \qquad \qquad \tilde{y}_1' = -2\tilde{y}_2$$
$$\text{and}$$
$$y_2' = -8y_1 \qquad \qquad \tilde{y}_2' = -8\tilde{y}_1$$

where $y_1 = \tilde{y}_1$ and $y_2 = -1 + \tilde{y}_2$. At $(0, 0)$ the system has a center and at $(0, -1)$ a saddle point.

CHAPTER 4 Series Solutions of Differential Equations Special Functions

Changes

This chapter has been streamlined and shortened by presenting the material on Bessel functions in a more condensed form and several minor changes to make it more teachable, without losing the opportunity to familiarize the student with an overview of some of the techniques used in connection with higher special functions.

SECTION 4.1. Power Series Method, page 194

Purpose. A simple introduction to the technique of the power series method in terms of simple examples whose solution the student knows very well.

SECTION 4.2. Theory of the Power Series Method, page 198

Purpose. Review of power series and a statement of the basic existence theorem for power series solutions (without proof, which would exceed the level of our presentation).

Main Content, Important Concepts

Radius of convergence (7)

Differentiation, multiplication of power series

Technique of index shift

Real analytic function (needed again in Sec. 4.4)

Comment

Depending on the preparation of the class, skip the section or discuss just a few less known facts.

SOLUTIONS TO PROBLEM SET 4.2, page 204

2. $y = a_0(4 - x^2 - \frac{1}{3}x^3 + \frac{1}{30}x^5 + \frac{1}{72}x^6 + \cdots)$

4. $y = a_0(1 + x^2 + x^4 + \cdots) = a_0/(1 - x^2)$

6. $y = (a_0 + a_1x)\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) = (a_0 + a_1x)e^{x^2}$

8. $y = a_1x + a_0(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \frac{1}{7}x^8 - \cdots)$. [This is a particular case of Legendre's equation ($n = 1$), which we consider in Sec. 4.3.]

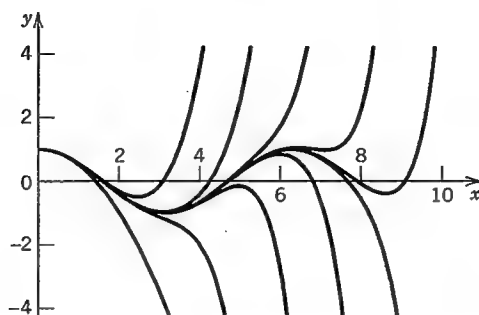
10. $y = a_0\left(1 + \frac{t^2}{2!} + \cdots\right) + a_1\left(t + \frac{t^3}{3!} + \cdots\right) = A\cosh x + B\sinh x$, where $A = a_0\cosh 1 - a_1\sinh 1$, $B = a_1\cosh 1 - a_0\sinh 1$.

12. The answer to Prob. 11 shows that the equation does not have a solution in powers of x, because of $\ln x$. The reason is that the coefficient $1/x$ of the equation is not analytic at $x = 0$. If we substitute a power series in powers of x into $xy' = y + x$, we get $a_0 = 0$, $a_1 = a_1 + 1$, a contradiction.

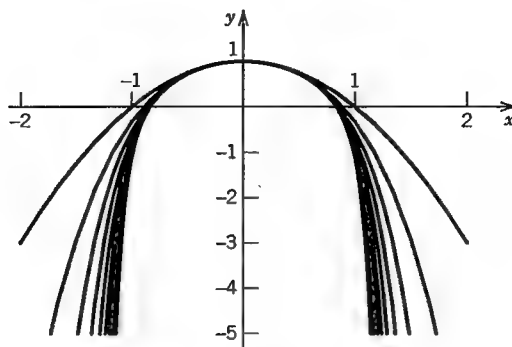
14. $R = 1$ **16.** $R = 1$ **18.** $R = \infty$ **20.** $R = 0$

22. TEAM PROJECT. The student should see that power series reveal many basic properties of the functions that they represent. The familiarity with the functions considered should help students understand the basic idea without being irritated by unfamiliar notions or notations and more involved formulas. Some of the tasks in (d) illustrate that not *all* properties become immediately visible, although all of them are determined by the sequence of the coefficients.

24. $\sum_{m=0}^{\infty} (m+2)(m+1)x^m, \quad 1$

26. CAS PROJECT. **(a)** It is instructive to see how polynomials of increasing degree follow more and more the cosine curve and then at a distinctly noticeable point begin to go their own way (see the figure). Some calculus books also show this, but students may have forgotten, so this reminder serves a good purpose. Those "qualitative" break-away points are very roughly at 1, 2, 3, · · · . Of course, for quantitative information, one would need more exact analytical estimates of remainders.

Section 4.2. CAS Project 26(a)

(b) The plot in the figure suggests that all the partial sums are even functions and that convergence seems to take place for $-1 < x < 1$; of course, this does not prove that the convergence radius is 1. Divide the equation by the coefficient of y'' to see that we cannot expect convergence in a larger interval because $1 - x^2$ is zero at $x = \pm 1$. The series solution is

$$y = 1 - \sum_{m=1}^{\infty} \frac{x^{2m}}{2m-1}.$$

Section 4.2. CAS Project 26(b)

SECTION 4.3. Legendre's Equation. Legendre Polynomials $P_n(x)$, page 205

Purpose. This section on Legendre's equation, one of the most important equations, and its solutions is more than just an exercise on the power series method. It should give the student a feeling for the usefulness of power series in exploring properties of **special functions** and for the wealth of relations between functions of a one-parameter family (with parameter n).

Legendre's equation occurs again in Secs. 4.7, 4.8, and 11.11.

Comment on Literature and History

For literature on Legendre's equation and its solutions, see Refs. [1], [6], [11].

Legendre's work on the subject appeared in 1785 and Rodrigues's contribution (see Prob. 6) in 1816.

SOLUTIONS TO PROBLEM SET 4.3, page 209

6. We have

$$(x^2-1)^n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (x^2)^{n-m}.$$

Differentiating n times, we can express the product of occurring factors $(2n-2m)(2n-2m-1)\cdots$ as the quotient of factorials and get

$$\frac{d^n}{dx^n}[(x^2-1)^n] = \sum_{m=0}^{M} (-1)^m \frac{n!}{m!(n-m)!} \frac{(2n-2m)!}{(n-2m)!} x^{n-2m}$$

with M as in (11). Divide by $n!2^n$. Then the left side equals the right side in Rodrigues's formula and the right side equals the right side of (11).

10. TEAM PROJECT. (a) Following the hint, we obtain

$$\text{(A)}\qquad (1-2xu+u^2)^{-1/2} = 1 + \frac{1}{2}(2xu-u^2) + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}(2xu-u^2)^n + \cdots$$

and for the general term on the right

$$\text{(B)}\qquad (2xu-u^2)^m = (2x)^m u^m - m(2x)^{m-1}u^{m+1} + \frac{m(m-1)}{2!}(2x)^{m-2}u^{m+2} + \cdots.$$

Now u^n occurs in the first term of the expansion (B) of $(2xu-u^2)^n$, in the second term of the expansion (B) of $(2xu-u^2)^{n-1}$, and so on. From (A) and (B) we see that the coefficients of u^n in those terms are

$$\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}(2x)^n = a_n x^n \qquad \text{[see (9)]},$$

$$-\frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n-2)}(n-1)(2x)^{n-2} = -\frac{2n}{2n-1}\frac{n-1}{4}a_n x^{n-2} = a_{n-2}x^{n-2} \qquad \text{[see (9*)]},$$

and so on. This proves the assertion.

(b) Set $u = r_1/r_2$ and $x = \cos\theta$.

(c) Use the formula for the sum of the geometric series and set $x = 1$ and $x = -1$. Then set $x = 0$ and use

$$(1 + u^2)^{-1/2} = \sum \binom{-1/2}{m} u^{2m}.$$

(d) Abbreviate $1 - 2xu + u^2 = U$. Differentiation of (13) gives

$$-\frac{1}{2}U^{-3/2}(-2x + 2u) = \sum_{n=0}^{\infty} nP_n(x)u^{n-1}.$$

Multiply this equation by U and represent $U^{-1/2}$ by (13):

$$(x - u)\sum_{n=0}^{\infty} P_n(x)u^n = (1 - 2xu + u^2)\sum_{n=0}^{\infty} nP_n(x)u^{n-1}.$$

In this equation, u^n has the coefficients

$$xP_n(x) - P_{n-1}(x) = (n + 1)P_{n+1}(x) - 2nxP_n(x) + (n - 1)P_{n-1}(x).$$

Simplifying gives the asserted Bonnet recursion.

12. $P_1^{\,1}(x) = \sqrt{1 - x^2}$, $P_2^{\,1}(x) = 3x\sqrt{1 - x^2}$, $P_2^{\,2}(x) = 3(1 - x^2)$, $P_4^{\,2}(x) = (1 - x^2)(105x^2 - 15)/2$

SECTION 4.4. Frobenius Method, page 211

Purpose. To introduce the student to the Frobenius method (an extension of the power series method), which is important for equations with coefficients that have singularities, notably Bessel's equation, so that the power series method can no longer handle them. This extended method requires more patience and care.

Main Content, Important Concepts

Regular and singular points

Indicial equation, three cases of roots (one unexpected)

Frobenius theorem, forms of bases in those cases

Short Courses. Take a quick look at those bases in Frobenius's theorem, say how it fits with the Euler–Cauchy equation, and omit everything else.

Comment on "Regular Singular" and "Irregular Singular"

These terms are used in some books and papers, but there is hardly any need for confusing the student by using them, simply because we cannot do (and don't do) anything about "irregular singular points." A simple use of "regular" and "singular" (as in complex analysis, where holomorphic functions are also known as "regular analytic functions") may thus be the best terminology.

Comment on Footnote 11

Gauss was born in Braunschweig (Brunswick) in 1777. At the age of 16, in 1793 he discovered the method of least squares (Secs. 18.5, 23.9). From 1795 to 1798 he studied at Göttingen. In 1799 he obtained his doctor's degree at Helmstedt. In 1801 he published

his first masterpiece, *Disquisitiones arithmeticae* (*Arithmetical Investigations,* begun in 1795), thereby initiating modern number theory. In 1801 he became generally known when his calculations enabled astronomers (Zach, Olbers) to rediscover the planet Ceres, which had been discovered in 1801 but had been visible only very briefly. He became the director of the Göttingen observatory in 1807 and remained there until his death. In 1809 he published his famous *Theoria motus corporum coelestium in sectionibus conicis solem ambientium* (*Theory of the Heavenly Bodies Moving About the Sun in Conic Sections;* Dover Publications, 1963), resulting from his further work in astronomy. In 1814 he developed his method of numerical integration (Sec. 17.5). His *Disquisitiones generales circa superficies curvas* (*General Investigations Regarding Curved Surfaces,* 1828) represents the foundation of the differential geometry of surfaces and contributes to conformal mapping (Sec. 12.5). His clear conception of the complex plane dates back to his thesis, whereas his first publication on this topic was not before 1831. This is typical: Gauss left many of his most outstanding results (non-Euclidean geometry, elliptic functions, etc.) unpublished. His paper on the hypergeometric series published in 1812 is the first systematic investigation into the convergence of a series; it allows a study of many special functions from a single point of view.

SOLUTIONS TO PROBLEM SET 4.4, page 216

2. $y_1 = x + 1$, $y_2 = 1/(x + 1)$. *Check:* Set $x + 1 = z$ to get an Euler–Cauchy equation.

4. Substitution of (2) and the derivatives (2*) gives

$$\text{(A)}\quad 4\sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r-1} + 2\sum_{m=0}^{\infty} (m + r)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

Writing this out, we have

$$\begin{aligned} &4r(r - 1)a_0x^{r-1} + 4(r + 1)ra_1x^r + 4(r + 2)(r + 1)a_2x^{r+1} + \cdots \\ &\quad + 2ra_0x^{r-1} + 2(r + 1)\, a_1x^r + \qquad 2(r + 2)a_2x^{r+1} + \cdots \\ &\quad + \qquad a_0x^r + \qquad a_1x^{r+1} + \cdots = 0. \end{aligned}$$

By equating the sum of the coefficients of x^{r-1} to zero we obtain the indicial equation

$$4r(r - 1) + 2r = 0, \qquad \text{thus} \qquad r^2 - \tfrac{1}{2}r = 0.$$

The roots are $r_1 = \frac{1}{2}$ and $r_2 = 0$. This is Case 1.

By equating the sum of the coefficients of x^{r+s} in (A) to zero we obtain (take $m + r - 1 = r + s$, thus $m = s + 1$ in the first two series and $m = s$ in the last series)

$$4(s + r + 1)(s + r)a_{s+1} + 2(s + r + 1)a_{s+1} + a_s = 0.$$

By simplification we find that this can be written

$$4(s + r + 1)(s + r + \tfrac{1}{2})a_{s+1} + a_s = 0.$$

We solve this for a_{s+1} in terms of a_s:

$$\text{(B)} \qquad a_{s+1} = -\frac{a_s}{(2s+2r+2)(2s+2r+1)} \qquad (s = 0, 1, \cdots).$$

First solution. We determine a first solution $y_1(x)$ corresponding to $r_1 = \frac{1}{2}$. For $r = r_1$, formula (B) becomes

$$a_{s+1} = -\frac{a_s}{(2s+3)(2s+2)} \qquad (s = 0, 1, \cdots).$$

From this we get successively

$$a_1 = -\frac{a_0}{3 \cdot 2}, \qquad a_2 = -\frac{a_1}{5 \cdot 4}, \qquad a_3 = -\frac{a_2}{7 \cdot 6}, \qquad \text{etc.}$$

In many practical situations an explicit formula for a_m will be rather complicated. Here it is simple: by successive substitution we get

$$a_1 = -\frac{a_0}{3!}, \qquad a_2 = \frac{a_0}{5!}, \qquad a_3 = -\frac{a_0}{7!}, \qquad \cdots$$

and in general, taking $a_0 = 1$,

$$a_m = \frac{(-1)^m}{(2m+1)!} \qquad (m = 0, 1, \cdots).$$

Hence the first solution is

$$y_1(x) = x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^m = \sqrt{x}\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - + \cdots\right) = \sin\sqrt{x}.$$

Second solution. If you recognize y_1 as a familiar function, apply reduction of order (see Sec. 2.1). If not, start from (6) with $r_2 = 0$. For $r = r_2 = 0$, formula (B) [with A_{s+1} and A_s instead of a_{s+1} and a_s] becomes

$$A_{s+1} = -\frac{A_s}{(2s+2)(2s+1)} \qquad (s = 0, 1, \cdots).$$

From this we get successively

$$A_1 = -\frac{A_0}{2 \cdot 1}, \qquad A_2 = -\frac{A_1}{4 \cdot 3}, \qquad A_3 = -\frac{A_2}{6 \cdot 5},$$

and by successive substitution we have

$$A_1 = -\frac{A_0}{2!}, \qquad A_2 = \frac{A_0}{4!}, \qquad A_3 = -\frac{A_0}{6!}, \qquad \cdots$$

and in general, taking $A_0 = 1$,

$$A_m = \frac{(-1)^m}{(2m)!}.$$

Hence the second solution, of the form (6) with $r_2 = 0$, is

$$y_2(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^m = 1 - \tfrac{1}{2}x + \tfrac{1}{24}x^2 - + \cdots = \cos\sqrt{x}.$$

6. $y_1 = x^{-1}\cos 2x, \qquad y_2 = x^{-1}\sin 2x$

8. $y_1 = e^x, \qquad y_2 = e^x/x$

10. $y_1 = (x-1)^2, y_2 = 1/(x-1)^2$. *Check:* $z = x - 1$ gives an Euler–Cauchy equation.

12. $y_1 = 1 + \dfrac{x^2}{2^2} + \dfrac{x^4}{(2\cdot 4)^2} + \dfrac{x^6}{(2\cdot 4\cdot 6)^2} + \cdots,$

$$y_2 = y_1 \ln x - \frac{x^2}{4} - \frac{3x^4}{8\cdot 16} - \frac{11x^6}{64\cdot 6\cdot 36} - \cdots$$

14. $y_1 = \dfrac{x^2}{3} - \dfrac{x^4}{2\cdot 5} + \dfrac{x^6}{2^2 2!7} - \dfrac{x^8}{2^3 3!9} + - \cdots, \qquad y_2 = \dfrac{1}{x}$

16. TEAM PROJECT. **(b)** In (7b), Sec. 4.2,

$$\frac{a_{n+1}}{a_n} = \frac{(a+n)(b+n)}{(n+1)(c+n)} \to 1,$$

hence $R = 1$.

(c) In the second line,

$$\text{arc tan}\, x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + - \cdots \qquad (|x| < 1)$$

$$\text{arc sin}\, x = x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots \qquad (|x| < 1).$$

(d) The roots can be read from (15), brought to the form (1′) by multiplying it by x and dividing by $1 - x$; then $b_0 = c$ in (4) and $c_0 = 0$.

18. $y = AF(2, 2, 3; x) + Bx^{-2}$

20. $y = A(1 + 4x) + B\sqrt{x}\,F(-\tfrac{1}{2}, -\tfrac{3}{2}, \tfrac{3}{2}; x)$

SECTION 4.5. Bessel's Equation. Bessel Functions $J_\nu(x)$, page 218

Purpose. To derive the Bessel functions of the first kind J_ν and $J_{-\nu}$ by the Frobenius method. (This is a major application of that method.) To show that these functions constitute a basis if ν is not an integer, but are linearly dependent for integer $\nu = n$ (so that we must look later, in Sec. 4.6, for a second linearly independent solution). To show that various differential equations can be reduced to Bessel's equation (see Problem Set 4.5).

Main Content, Important Concepts

Derivation just mentioned

Linear independence of J_ν and $J_{-\nu}$ if ν is not an integer

Linear dependence of J_ν and $J_{-\nu}$ if $\nu = n = 1, 2, \cdots$

Gamma function as a tool

Short Courses. No derivation of any of the series. Discussion of J_0 and J_1 (which are similar to cosine and sine). Mention Theorem 2.

Comment on Special Functions

Since various institutions no longer find time to offer a course in special functions, Bessel functions may give another opportunity (together with Sec. 4.3) for getting at least some feeling for the flavor of the theory of special functions, which will continue to be of some significance to the engineer and physicist. For this reason we have added some material on basic relations for Bessel functions in this section.

SOLUTIONS TO PROBLEM SET 4.5, page 226

2. $y = AJ_\nu(\lambda x) + BJ_{-\nu}(\lambda x)$

From a practical point of view, this is probably the most frequently occurring case.

Problems 1–10 are for gaining skill and making the student aware of the fact that Bessel's equation, just as the hypergeometric equation in Problem Set 4.4, is a member of a large family of equations that can be solved in terms of Bessel functions, a fact that adds to the great importance of these functions.

4. $AJ_{1/6}(\sqrt{x}) + BJ_{-1/6}(\sqrt{x})$

6. $J_0(\sqrt{x})$

8. $\sqrt{x}\,[AJ_{1/2}(\sqrt{x}) + BJ_{-1/2}(\sqrt{x})] = x^{1/4}[\tilde{A}\sin\sqrt{x} + \tilde{B}\cos\sqrt{x}]$

10. $x^3J_3(x)$, and we do not get a general solution, by Theorem 2.

12. $$J_0' = \sum_{m=0}^{\infty} \frac{2m(-1)^m x^{2m-1}}{2^{2m}(m!)^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1}m!(m-1)!} = \sum_{s=0}^{\infty} \frac{(-1)^{s+1}x^{2s+1}}{2^{2s+1}(s+1)!s!},$$

where $m = s + 1$.

14. Use (7b), Sec. 4.2, and $$\frac{x^{2m+2}}{2^{2m+2+n}(m+1)!(n+m+1)!} \bigg/ \frac{x^{2m}}{2^{2m+n}m!(n+m)!} \to 0$$

as $m \to \infty$ (x fixed).

16. We obtain the following values. Note that the relative error of this very crude approximation is rather small.

x	Approximation	Exact (4D)	Relative Error (%)
0	1.0000	1.0000	0
0.1	0.9975	0.9975	0
0.2	0.9900	0.9900	0
0.3	0.9775	0.9776	0.01
0.4	0.9600	0.9604	0.04
0.5	0.9375	0.9385	0.1
0.6	0.9100	0.9120	0.2
0.7	0.8775	0.8812	0.4
0.8	0.8400	0.8463	0.7
0.9	0.7975	0.8075	1.2
1.0	0.7500	0.7652	2.0

18. Let $x > 0$. We have $J_0(x) = s_2(x) + R_2(x) = s_4(x) + R_4(x)$, $s_2(x) < J_0(x) < s_4(x)$, $s_2(x_0) < 0 < s_4(x_0)$; hence $x_1 < x_0 < x_2$, where $x_1 = 2$ and $x_2 = \sqrt{8}$ are defined by $s_2(x_1) = 0$, $s_4(x_2) = 0$.

20. $J_0' = 0$ at least once between two consecutive zeros of J_0, by Rolle's theorem. Now (25) with $\nu = 0$ is

$$J_0' = -J_1.$$

Together, J_1 has at least one zero between two consecutive zeros of J_0.

Furthermore, $(xJ_1)' = 0$ at least once between two consecutive zeros of xJ_1, hence of J_1 (also at $x = 0$ since $J_1(0) = 0$), by Rolle's theorem. Now (24) with $\nu = 1$ is

$$(xJ_1)' = xJ_0.$$

Together, J_0 has at least one zero between two consecutive zeros of J_1.

22. Integrate (24).

24. Integrate (27).

26. Integrate (24) with $\nu = 2$ to get

(a)
$$\int x^2 J_1\,dx = x^2 J_2 + c.$$

Integrate (24) with $\nu = 1$ to get

(b)
$$\int x J_0\,dx = x J_1 + c.$$

Integrating by parts, using (b), and again, using (a), we get

$$\begin{aligned}\int x^3 J_0\,dx &= \int x^2(xJ_0)\,dx \\ &= x^2(xJ_1) - 2\int x^2 J_1\,dx \\ &= x^3 J_1 - 2x^2 J_2 + c.\end{aligned}$$

28. TEAM PROJECT. Assuming small angles α in the displacement, we can regard $W(x)$ to be approximately equal to the tension acting tangentially in the moving cable. The restoring force is the horizontal component of the tension. For the difference in force we use the mean value theorem of differential calculus. By Newton's second law this equals the mass $\rho\Delta x$ times the acceleration u_{tt} of this portion of the cable. The substitution of u first gives

$$-\omega^2 y\cos(\omega t + \delta) = g[(L - x)y']'\cos(\omega t + \delta).$$

Now drop the cosine factor, perform the differentiation, and order the terms.

(b) $dx = -dz$ and by the chain rule,

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} + \lambda^2 y = 0.$$

In the next transformation the chain rule gives

$$\frac{dy}{dz} = \frac{dy}{ds}\lambda z^{-1/2}, \qquad \frac{d^2y}{dz^2} = \frac{d^2y}{ds^2}\lambda^2 z^{-1} - \frac{1}{2}\frac{dy}{ds}\lambda z^{-3/2}.$$

Substitution gives

$$\lambda^2\frac{d^2y}{ds^2} + \left(-\frac{1}{2}\lambda z^{-1/2} + \lambda z^{-1/2}\right)\frac{dy}{ds} + \lambda^2 y = 0.$$

Now divide by λ^2 and remember that $s = 2\lambda z^{1/2}$. This gives Bessel's equation.

(c) This follows from the fact that the upper end ($x = 0$) is fixed. The second normal mode looks similar to the portion of J_0 between the second positive zero and the origin. Similarly for the third normal mode. The first positive zero is about 2.405. For the cable of 2 meters this gives the frequency

$$\frac{\omega}{2\pi} = \frac{2.405}{2\cdot 2\pi\sqrt{L/g}} = \frac{2.405}{4\pi\sqrt{2.00/9.80}} = 0.424\ [\text{sec}^{-1}] = 25.4\ [\text{cycles/min}].$$

Similarly, we obtain 11.4 cycles/min for the long cable.

30. CAS PROJECT. **(b)** $x_0 = 1$, $x_1 = 2.5$, $x_2 = 20$, approximately. It increases with n.

(c) (14) is exact.

(d) It oscillates.

(e) Formula (25) with $\nu = 0$.

SECTION 4.6. Bessel Functions of the Second Kind, page 228

Purpose. Derivation of a second independent solution, which is still missing in the case of $\nu = n = 0, 1, \cdots$.

Main Content

Detailed derivation of $Y_0(x)$

Cursory derivation of $Y_n(x)$ for any n

General solution (9) valid for all ν, integer or not

Short Courses. Omit this section.

Comment on Hankel Functions and Modified Bessel Functions

These are included for completeness, but will not be needed in our further work.

SOLUTIONS TO PROBLEM SET 4.6, page 232

2. $AJ_5(x) + BY_5(x)$

4. Substitute $y = ux^{1/2}$ and its derivatives into the given equation and multiply the resulting equation by $x^{3/2}$ to get

$$x^2u'' + xu' + (x^3 - \tfrac{1}{4})u = 0.$$

Now introduce z as given in the problem statement to get the answer

$$y = \sqrt{x}\,[AJ_{1/3}(\tfrac{2}{3}x^{3/2}) + BJ_{-1/3}(\tfrac{2}{3}x^{3/2})].$$

6. $\sqrt{x}\,[AJ_{1/3}(\tfrac{2}{3}kx^{3/2}) + BY_{1/3}(\tfrac{2}{3}kx^{3/2})]$. For $k = i = \sqrt{-1}$, this equation is called **Airy's equation.** Its solutions (*"Airy functions"*) have been extensively investigated; for some formulas and graphs, see M. Abramowitz and I. A. Stegun [1], pp. 446–52, listed in Appendix 1.

8. $\sqrt{x}\,[AJ_{1/6}(\tfrac{1}{3}kx^3) + BY_{1/6}(\tfrac{1}{3}kx^3)]$

10. $x^\nu[AJ_\nu(x^\nu) + BJ_{-\nu}(x^\nu)]$

12. Approximate values $\pi/4 = 0.79$, $5\pi/4 = 3.93$, $9\pi/4 = 7.07$

14. Use (20) in Sec. 4.5.

16. Since I_ν is a solution of (12), so is $I_{-\nu}$ because (12) involves ν^2 and is linear and homogeneous. Hence K_ν is a solution of (12).

The problem illustrates that for different purposes different special functions were introduced and investigated. It would lead us too far to show applications where those K_ν are of practical advantage. See Watson's standard treatise [A7] in Appendix 1.

SECTION 4.7. Sturm–Liouville Problems. Orthogonal Functions, page 233

Purpose. Discussion of eigenvalue problems for ordinary second-order differential equations (1) under boundary conditions (2).

Main Content, Important Concepts

Sturm–Liouville equations, Sturm–Liouville problem

Reality of eigenvalues

Orthogonality of eigenfunctions

Orthogonality of Legendre polynomials and Bessel functions

Short Courses. Omit this section.

Comment on Importance

This theory owes its significance to two factors. On the one hand, boundary value problems involving practically important equations (Legendre's, Bessel's, etc.) can be cast into Sturm–Liouville form, so that here we have a general theory with several important particular cases. On the other hand, the theory gives important general results on the spectral theory of those problems.

Comment on Existence of Eigenvalues

This theory is difficult. Quite generally, in problems where we can have *infinitely many* eigenvalues, the existence problem becomes nontrivial, in contrast to matrix eigenvalue problems (Chap. 7), where existence is trivial, a consequence of the fact that a polynomial equation $f(x) = 0$ (f not constant) has at least one solution and at most n numerically different ones (n the degree of the polynomial).

SOLUTIONS TO PROBLEM SET 4.7, page 238

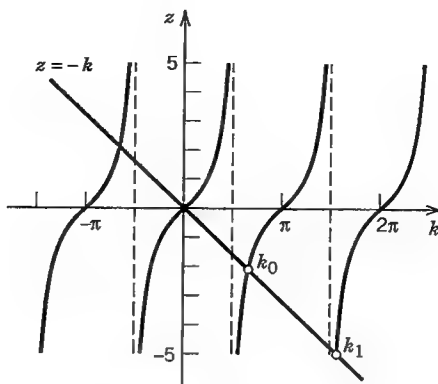
2. If y_m is a solution of (1), so is z_m because (1) is linear and homogeneous; here, $\lambda = \lambda_m$, the eigenvalue corresponding to y_m. Also, multiplying (2) with $y = y_m$ by c, we see that z_m also satisfies the boundary conditions. This proves the assertion.

4. $\lambda = (n\pi/L)^2$, $n = 1, 2, \cdots$; $y_n(x) = \sin(n\pi x/L)$

6. $\lambda = n^2$, $n = 0, 1, 2, \cdots$; $y_0(x) = 1$, $y_n(x) = \cos nx, \sin nx$ $(n \geqq 1)$

8. $\lambda = n^2$, $n = 1, 2, \cdots$; $y_n(x) = e^{-x} \sin nx$

10. $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(\pi) = 0$

12. The k_n are obtained as intersections of $z = \tan k$ and $z = -k$; see the figure. $k_0 \approx 2.029$, $k_1 \approx 4.913$, approximately.

Section 4.7. Problem 12

14. $a = -\pi, b = \pi, c = \pi, k = 0$

16. TEAM PROJECT. (a) We integrate over x from -1 to 1, hence over θ defined by $x = \cos\theta$ from π to 0. Using $(1 - x^2)^{-1/2}\,dx = -d\theta$, we thus obtain

$$\int_{-1}^{1} \cos(m \arccos x)\cos(n \arccos x)(1 - x^2)^{-1/2}\,dx$$

$$= \int_0^{\pi} \cos m\theta \cos n\theta\, d\theta = \frac{1}{2}\int_0^{\pi} (\cos(m+n)\theta + \cos(m-n)\theta)\, d\theta,$$

which is zero for integer $m \neq n$.

(b) Following the hint, we calculate $\int e^{-x}x^k L_n\,dx = 0$ for $k < n$:

$$\int_0^{\infty} e^{-x}x^k L_n(x)\,dx = \frac{1}{n!}\int_0^{\infty} x^k \frac{d^n}{dx^n}(x^n e^{-x})\,dx = -\frac{k}{n!}\int_0^{\infty} x^{k-1}\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\,dx$$

$$= \cdots = (-1)^k \frac{k!}{n!}\int_0^{\infty} \frac{d^{n-k}}{dx^{n-k}}(x^n e^{-x})\,dx = 0.$$

SECTION 4.8. Orthogonal Eigenfunction Expansions, page 240

Purpose. To show how families (sequences) of orthogonal functions, as they arise in eigenvalue problems and elsewhere, are used in series for representing other functions, and to show how orthogonality becomes crucial in simplifying the determination of the coefficients of such a series by integration.

Main Content, Important Concepts

Standard notation (y_m, y_n)

Orthogonal expansion (3), eigenfunction expansion

Fourier constants (4)

Fourier series (5), Euler formulas (6)

Short Courses. Omit this section.

Comment on Flexibility on Fourier Series

Since Sec. 4.8, with the definition of orthogonality taken from Sec. 4.7 and Examples 2 and 3 omitted, is independent of other sections in this chapter, it could also be used after Chap. 10 on Fourier series. We did not put it there for reasons of time and because Chap. 10 is intimately related to the main applications of Fourier series (to partial differential equations) in Chap. 11.

Comment on Notation

(y_m, y_n) is not a must, but has become standard; perhaps if it is written out a few times, it will stop irritating poorer students.

SOLUTIONS TO PROBLEM SET 4.8, page 246

2. By (7), where $f(x)$ is the given polynomial, or by undetermined coefficients, starting from

$$f(x) = a_0P_0 + a_1P_1(x) + a_2P_2(x) + a_3P_3(x)$$

and equating coefficients of like powers on both sides, we get

$$f(x) = -4P_0 + 2P_1 - 4P_2 + 8P_3.$$

4. $x^2 = \frac{1}{3}P_0 + \frac{2}{3}P_2, \quad x^3 = \frac{3}{5}P_1 + \frac{2}{5}P_3, \quad x^4 = \frac{1}{5}P_0 + \frac{4}{7}P_2 + \frac{8}{35}P_4$

6. $e^x = a_0 + a_1P_1(x) + a_2P_2(x) + \cdots$, where, by (7), and by (11′) in Sec. 4.3,

$$a_0 = \frac{1}{2}\int_{-1}^{1} e^x\,dx = \sinh 1 = 1.1752$$

$$a_1 = \frac{3}{2}\int_{-1}^{1} xe^x\,dx = \frac{3}{2}(x-1)e^x\Big|_{-1}^{1} = 3e^{-1} = 1.1036$$

$$a_2 = \frac{5}{2}\int_{-1}^{1}\frac{1}{2}(3x^2-1)e^x\,dx = \frac{5}{4}\left[(3x^2-1)e^x\Big|_{-1}^{1} - 6\int_{-1}^{1} xe^x\,dx\right]$$

$$= \frac{5}{4}\left[2(e-e^{-1}) - 6(x-1)e^x\Big|_{-1}^{1}\right]$$

$$= 5\sinh 1 - 15e^{-1} = 0.3578, \text{ etc.}$$

Answer:

$$e^x = 1.1752P_0 + 1.1036P_1 + 0.3578P_2 + 0.0705P_3 + \cdots.$$

8. From (7) we obtain

$$f(x) = 0.5P_0 - 0.9375P_2 + 0.5273P_4 + 0.1333P_6 - 0.4910P_8 + \cdots.$$

10. TEAM PROJECT. (b) A Maclaurin series $f(t) = \sum_{n=0}^{\infty} a_nt^n$ has the coefficients $a_n = f^{(n)}(0)/n!$. We thus obtain

$$f^{(n)}(0) = \frac{d^n}{dt^n}(e^{tx-t^2/2})\Big|_{t=0} = e^{x^2/2}\frac{d^n}{dt^n}(e^{-(x-t)^2/2})\Big|_{t=0}.$$

If we set $x - t = z$, this becomes

$$f^{(n)}(0) = e^{x^2/2}(-1)^n\frac{d^n}{dz^n}(e^{-z^2/2})\Big|_{z=x} = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}(e^{-x^2/2}) = He_n(x).$$

(c) $G_x = \sum a_n'(x)t^n = \sum He_n'(x)t^n/n! = tG = \sum He_{n-1}(x)t^n/(n-1)!$, etc.

(d) We write $e^{-x^2/2} = v$, $v^{(n)} = d^nv/dx^n$, etc., and use (21). By integrations by parts, for $n > m$,

$$\int_{-\infty}^{\infty} vHe_mHe_n\,dx = (-1)^n\int_{-\infty}^{\infty} He_mv^{(n)}\,dx = (-1)^{n-1}\int_{-\infty}^{\infty} He_m'v^{(n-1)}\,dx$$

$$= (-1)^{n-1}m\int_{-\infty}^{\infty} He_{m-1}v^{(n-1)}\,dx = \cdots$$

$$= (-1)^{n-m}m!\int_{-\infty}^{\infty} He_0v^{(n-m)}\,dx = 0.$$

(e) $nHe_n = nxHe_{n-1} - nHe_{n-1}'$ from (22) with $n-1$ instead of n. In this equation, the first term on the right equals xHe_n' by (21). The last term equals $-He_n''$, as follows by differentiation of (21).

We write $y = Ew$, where $E = e^{x^2/4}$. Then

$$y' = \tfrac{1}{2}xEw + Ew'$$
$$y'' = \tfrac{1}{2}Ew + \tfrac{1}{4}x^2Ew + xEw' + Ew''.$$

Substitute this into the differential equation (23) and divide by E to get the result. The point is that the new equation does not contain a first derivative; hence our transformation is precisely that for eliminating the first derivative from (23).

SOLUTIONS TO CHAPTER 4 REVIEW, page 247

16. $(x - 2)^2$, $(x - 2)^{-3}$. This is an Euler–Cauchy equation with independent variable $t = x - 2$.

18. e^{-x^2}, xe^{-x^2}

20. $x^{-2} \cos (x^2)$, $x^{-2} \sin (x^2)$

22. e^x, $e^x \ln x$

24. x, $x \ln x + x^2$

26. $1/\sqrt{\pi}$, $\sqrt{2/\pi} \cos nx$, $n = 1, 2, \cdots$

28. $\dfrac{1}{\sqrt{2}}, \sqrt{\dfrac{3}{2}}\, x, \sqrt{\dfrac{5}{8}}\,(3x^2 - 1), \sqrt{\dfrac{7}{8}}\,(5x^3 - 3x)$

30. $\lambda = n^2$, $y_n = \sin nx$, $n = 1, 2, \cdots$

32. $\lambda = \alpha_{n1}$ = the nth positive zero of $J_1(x)$, $n = 1, 2, \cdots$, $y_n = J_1(\alpha_{n1}x)$

34. $\frac{3}{7}P_1 + \frac{4}{9}P_3 + \frac{8}{63}P_5$, $\frac{1}{7}P_0 + \frac{10}{21}P_2 + \frac{24}{77}P_4 + \frac{16}{231}P_6$

36. $-16P_6\,(x)$

38. $AJ_\nu(2x) + BY_\nu(2x)$

40. $AJ_{1/4}(x) + BJ_{-1/4}(x)$

CHAPTER 5 Laplace Transforms

Major Changes

The first shifting theorem has been moved ahead to Sec. 5.1, where it fits much better and helps to simplify the presentation. Further streamlining has been achieved by placing the unit step function and Dirac's delta in the same section (Sec. 5.3). The impractical theoretical formulas for the Laplace transforms of partial fractions have been replaced by a more practical approach in terms of key examples related to differential equations (Sec. 5.6). The application of the Laplace transform to systems of differential equations is discussed in the new Sec. 5.7.

SECTION 5.1. Laplace Transform. Inverse Transform. Linearity. Shifting, page 251

Purpose. To explain the basic concepts, to present a short list of basic transforms, and to show how these are derived from the definition.

Main Content, Important Concepts

Transform, inverse, linearity

First shifting theorem

Table 5.1

Existence and its practical significance

Comment on Table 5.1

After working for a while in this chapter, the student should be able to memorize these transforms. Further transforms in Sec. 5.9 are derived as we go along, many of them from Table 5.1.

SOLUTIONS TO PROBLEM SET 5.1, page 257

2. $a/s + b/s^2 + 2c/s^3$

4. $\cos^2 \omega t = \frac{1}{2} + \frac{1}{2}\cos 2\omega t$; transform $1/2s + s/(2s^2 + 8\omega^2)$

6. $e^t \cosh 3t = \frac{1}{2}(e^{4t} + e^{-2t})$; transform

$$\frac{1}{2}\left(\frac{1}{s-4} + \frac{1}{s+2}\right) = \frac{s-1}{s^2 - 2s - 8}$$

8. $\sin 2t \cos 2t = 2 \sin 4t$; transform $2/(s^2 + 16)$

10. $\dfrac{k}{s}(e^{-s} - e^{-4s})$

12. $\dfrac{(1 - e^{-s})^2}{s^2}$

14. $\dfrac{1 - e^{-s}}{s^2} - \dfrac{1}{se^{2s}}$

16. $\dfrac{b}{as^2}(1 - e^{-as}) - \dfrac{b}{s}e^{-as}$

18. $5 \cosh 5t$

20. $\cosh 2t - 2 \sinh 2t$

22. $\frac{1}{2}t^2 + \frac{1}{4}t^4 + \frac{1}{12}t^6$

24. $e^t + e^{-2t} - 2e^{3t}$

26. $t^3 + e^{3t}$

28. $\dfrac{2s^3}{s^4 - 1} = \dfrac{s}{s^2 + 1} + \dfrac{s}{s^2 - 1}$. *Answer:* $\cos t + \cosh t$

30. $\dfrac{s + \alpha}{(s + \alpha)^2 + \beta^2}$

32. $\dfrac{1}{s + 1} + \dfrac{s + 1}{s^2 + 2s + 2}$

34. $\dfrac{2}{(s - 1)^3} + \dfrac{2}{(s - 1)^2} + \dfrac{1}{s - 1}$

36. $2t^3e^{3t}$

38. $2e^t \sinh 2t = e^{3t} - e^{-t}$

40. $4e^{-t/2} \sin \frac{1}{2}t$

42. Let $f = \mathcal{L}^{-1}(F)$, $g = \mathcal{L}^{-1}(G)$. Since the *transform* is linear, we obtain

$$aF + bG = a\mathcal{L}(f) + b\mathcal{L}(g) = \mathcal{L}(af + bg).$$

Now apply $\mathcal{L}^{-1}$ on both sides to get the desired result,

$$\mathcal{L}^{-1}(aF + bG) = \mathcal{L}^{-1}\mathcal{L}(af + bg) = af + bg = a\mathcal{L}^{-1}(F) + b\mathcal{L}^{-1}(G).$$

Note that we have proved much more than just the claim, namely, the **theorem:** *If a linear transformation has an inverse, the inverse is linear.*

44. We first use the definition (1). Then we set $ct = v$, so that

$$t = \frac{v}{c}, \qquad dt = \frac{dv}{c}, \qquad -st = -\frac{sv}{c} = -\left(\frac{s}{c}\right) v.$$

Thus,

$$\mathcal{L}(f(ct)) = \int_0^\infty e^{-st} f(ct)\, dt = \int_0^\infty e^{-(s/c)v} f(v) \frac{dv}{c} = \frac{1}{c} F\left(\frac{s}{c}\right).$$

The application is straightforward, with $c = \omega$.

SECTION 5.2. Transforms of Derivatives and Integrals. Differential Equations, page 258

Purpose. To get a first impression of how the Laplace transform solves ordinary differential equations and initial value problems, the task for which it is designed.

Main Content, Important Concepts

(1) $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$

Extension of (1) to higher derivatives [(2)–(4)]

Solution of a differential equation, subsidiary equation

Transfer function

Transform of the integral of a function

Shifted data problems

Comment on Differential Equations

The last of the three steps of solution is the hardest, but we shall derive many general properties of the Laplace transform (collected in Sec. 5.8) that will help, along with formulas in Table 5.1 and those in Sec. 5.9, so that we can proceed to equations for which the present method is superior to the classical one.

SOLUTIONS TO PROBLEM SET 5.2, page 264

2. $y = \frac{7}{6}e^{5t} - \frac{1}{6}e^{-4t}$

4. $y = 3e^{-t} + 5e^{2t}$

6. $y = 3\cos t + (4 + t)\sin t$

8. $y = -25 + 0.5t^2$

10. PROJECT. **(b)** Theorems 1 and 2 are more important because they are crucial in solving differential equations, whereas Theorem 3 serves as a tool for obtaining new transforms.

(c) In the integration by parts shown in the proof of Theorem 1 we now have to integrate from 0 to a and then from a to ∞, thus obtaining $f(a - 0)e^{-as}$ from the upper limit of integration of the first integral and $-f(a + 0)e^{-as}$ from the lower limit of integration of the second integral.

(d) For the given function, $f(2 + 0) - f(2 - 0) = -1$, $f(0) = 0$, so that (1*) and $\mathcal{L}(f') = (1 - e^{-s})/s$ give

$$\mathcal{L}(f) = (1 - e^{-s} - se^{-2s})/s^2.$$

12. PROJECT. We derive (a). We have $f(0) = 0$ and

$$f'(t) = \cos \omega t - \omega t \sin \omega t, \qquad f'(0) = 1$$
$$f''(t) = -2\omega \sin \omega t - \omega^2 f(t).$$

By (2),

$$\mathcal{L}(f'') = -2\omega \frac{\omega}{s^2 + \omega^2} - \omega^2 \mathcal{L}(f) = s^2 \mathcal{L}(f) - 1.$$

Collecting $\mathcal{L}(f)$-terms, we obtain

$$\mathcal{L}(f)(s^2 + \omega^2) = \frac{-2\omega^2}{s^2 + \omega^2} + 1 = \frac{s^2 - \omega^2}{s^2 + \omega^2}.$$

Division by $s^2 + \omega^2$ on both sides gives (a).

In (b) on the right we get from (a)

$$\mathcal{L}(\sin \omega t - \omega t \cos \omega t) = \frac{\omega}{s^2 + \omega^2} - \omega \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}.$$

Taking the common denominator and simplifying the numerator,

$$\omega(s^2 + \omega^2) - \omega(s^2 - \omega^2) = 2\omega^3$$

gives (b).

(c) is shown in Example 4.

(d) is derived the same way as (b), with + instead of −, so that the numerator is

$$\omega(s^2 + \omega^2) + \omega(s^2 - \omega^2) = 2\omega s^2,$$

which gives (d).

(e) is similar to (a). We have $f(0) = 0$ and obtain

$$f'(t) = \cosh at + at \sinh at, \qquad f'(0) = 1$$
$$f''(t) = 2a \sinh at + a^2 f(t).$$

By (2) we obtain

$$\mathcal{L}(f'') = \frac{2a^2}{s^2 - a^2} + a^2\mathcal{L}(f) = s^2\mathcal{L}(f) - 1.$$

Hence

$$\mathcal{L}(f)(s^2 - a^2) = \frac{2a^2}{s^2 - a^2} + 1 = \frac{s^2 + a^2}{s^2 - a^2}.$$

Division by $s^2 - a^2$ gives (e).

(f) follows similarly. We have $f(0) = 0$ and, furthermore,

$$f'(t) = \sinh at + at\cosh at, \qquad f'(0) = 0$$

$$f''(t) = 2a\cosh at + a^2 f(t)$$

$$\mathcal{L}(f''(t)) = 2a\frac{s}{s^2 - a^2} + a^2\mathcal{L}(f) = s^2\mathcal{L}(f)$$

$$\mathcal{L}(f)(s^2 - a^2) = \frac{2as}{s^2 - a^2}.$$

Division by $s^2 - a^2$ gives formula (f).

14. $e^{2t} - 2t - 1$ **16.** $\cos t + \frac{1}{2}t^2 - 1$

18. $2 - t - 2e^{-t}$ **20.** $\sin \pi t + \frac{1}{6}\pi^3 t^3 - \pi t$

SECTION 5.3. Unit Step Function. Second Shifting Theorem. Dirac's Delta Function, page 265

Purpose

1. To introduce the unit step function $u(t - a)$, which together with Dirac's delta greatly increases the usefulness of the Laplace transform.
2. To find the transform of

$$0 \quad (t < a), \qquad f(t - a) \quad (t > a)$$

if that of $f(t)$ is known (*"t-shifting"*). (*"s-shifting"* was considered in Sec. 5.1.)

3. To model short impulses by Dirac's delta $\delta(t - a)$.

Main Content, Important Concepts

Unit step function (1), its transform (4)

Second shifting theorem (Theorem 1)

Dirac's delta, its transform (8)

Comment on the Unit Step Function

Problem Set 5.3 shows that $u(t - a)$ is the basic function for representing discontinuous functions.

SOLUTIONS TO PROBLEM SET 5.3, page 273

2. The representation needed for applying the second shifting theorem is

$$tu(t - 1) = (t - 1)u(t - 1) + u(t - 1)$$

and gives the transform

$$e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right).$$

4. $2e^{-s}/s^3$

6. $e^{-2t}u(t-3) = e^{-2(t-3)}e^{-6}u(t-3)$. *Answer:* $e^{-3s-6}/(s+2)$

8. $t^2(1-u(t-1)) = t^2 - [(t-1)^2 + 2(t-1) + 1]u(t-1)$.
Answer: $2s^{-3} - e^{-s}(2s^{-3} + 2s^{-2} + s^{-1})$

10. $(1-e^{-t})[1-u(t-2)] = 1 - e^{-t} - (1 - e^{-(t-2)}e^{-2})u(t-2)$.
Answer: $\frac{1}{s} - \frac{1}{s+1} - e^{-2s}\left(\frac{1}{s} - \frac{e^{-2}}{s+1}\right)$

12. The given function is

$$[u(t-2\pi) - u(t-4\pi)]\sin t = u(t-2\pi)\sin(t-2\pi) - u(t-4\pi)\sin(t-4\pi),$$

so that we get the *answer*

$$\frac{1}{s^2+1}(e^{-2\pi s} - e^{-4\pi s}).$$

14. $4u(t-2) - 8u(t-5)$

16. s^{-3} has the inverse $t^2/2$, hence $(s-1)^{-3}$ has the inverse $e^t t^2/2$ (first shifting), and $e^{-3s}/(s-1)^3$ has the inverse $\frac{1}{2}e^{t-3}(t-3)^2u(t-3)$ (second shifting).

18. $s^2 + 2s + 2 = (s+1)^2 + 1$. Hence the reciprocal of this has the inverse $e^{-t}\sin t$, and the second shifting theorem gives the *answer* $e^{-(t-2\pi)}(\sin t)u(t-2\pi)$.

20. $y = 3e^{t/2}(\cos 3t + \sin 3t)$

22. In terms of unit step functions the function on the right is

$$r(t) = 4t[1-u(t-1)] + 8u(t-1) = 4t - [4(t-1) - 4]u(t-1).$$

Answer:

$$y = \begin{cases} 4e^{-t} - e^{-2t} + 2t - 3 & \text{if } 0 < t < 1 \\ (4-8e)e^{-t} + (3e^2-1)e^{-2t} + 4 & \text{if } \quad t > 1. \end{cases}$$

24. $r(t) = 4e^t[1-u(t-2)] = 4e^t - 4e^2e^{(t-2)}u(t-2)$. Hence

$$s^2Y - s + 2 - 5(sY-1) + 6Y = \frac{4}{s-1} - \frac{4e^2e^{-2s}}{s-1}.$$

Take $-s+7$ to the right, divide by $s^2 - 5s + 6 = (s-2)(s-3)$ to get

$$Y = \frac{s-7}{(s-2)(s-3)} + \frac{4}{(s-1)(s-2)(s-3)} - \frac{4e^2e^{-2s}}{(s-1)(s-2)(s-3)}.$$

The sum of the first two terms on the right has the partial fraction expansion

$$\frac{2}{s-1} + \frac{1}{s-2} - \frac{2}{s-3}, \qquad \text{hence the inverse} \qquad 2e^t + e^{2t} - 2e^{3t};$$

this is the solution if $0 < t < 2$. For $t > 2$ the solution equals the solution just given plus the inverse of

$$-\frac{4e^2e^{-2s}}{(s-1)(s-2)(s-3)} = e^2e^{-2s}\left(-\frac{2}{s-1} + \frac{4}{s-2} - \frac{2}{s-3}\right);$$

this inverse is

$$e^2[-2e^{t-2} + 4e^{2(t-2)} - 2e^{3(t-2)}]u(t-2).$$

The sum of this and the previous solution is

$$(1 + 4e^{-2})e^{2t} + (-2 - 2e^{-4})e^{3t};$$

this is the solution if $t > 2$.

26. $y = 2\cos 4t + u(t - \pi)\sin 4(t - \pi)$
$= 2\cos 4t$ if $0 < t < \pi$, and $2\cos 4t + \sin 4t$ if $t > \pi$

28. $y = 3e^{-2t}\sin t\ (0 < t < 1)$, $y = e^{-2t}[3\sin t + e^2 \sin(t - 1)]\ (t > 1)$

30. $(s + 2)(s + 3)Y = e^{-s}/s + e^{-2s} + 1$. Use

$$\frac{1}{s(s+2)(s+3)} = \frac{1/6}{s} - \frac{1/2}{s+2} + \frac{1/3}{s+3}.$$

Answer:

$$y = e^{-2t} - e^{-3t} \qquad \text{if } 0 < t < 1,$$

$$y = e^{-2t} - e^{-3t} + \tfrac{1}{6} - \tfrac{1}{2}e^{-2(t-1)} + \tfrac{1}{3}e^{-3(t-1)}$$
$$= \tfrac{1}{6} + (1 - \tfrac{1}{2}e^2)e^{-2t} - (1 - \tfrac{1}{3}e^3)e^{-3t} \qquad \text{if } 1 < t < 2,$$

$$y = \tfrac{1}{6} + (1 - \tfrac{1}{2}e^2)e^{-2t} - (1 - \tfrac{1}{3}e^3)e^{-3t} + e^{-2(t-2)} - e^{-3(t-2)}$$
$$= \tfrac{1}{6} + (1 - \tfrac{1}{2}e^2 + e^4)e^{-2t} - (1 - \tfrac{1}{3}e^3 + e^6)e^{-3t} \qquad \text{if } t > 2.$$

32. $(-L\cos t + R\sin t + e^{-Rt/L})/(L^2 + R^2)$ if $0 < t < 2\pi$,
$Le^{-Rt/L}(1 - e^{2\pi R/L})/(L^2 + R^2)$ if $t > 2\pi$

34. $v = 1 - u(t - a)$. Subsidiary equation:

$$sI + I/s = 1/s - e^{-as}/s.$$

Answer:

$$i = \begin{cases} \sin t & \text{if } \ 0 < t < a \\ \sin t - \sin(t - a) & \text{if } \ t > a. \end{cases}$$

36. $I = 100(e^{-s} - e^{-1.01s})/(s + 0.1)$, $\quad i = 0 \quad$ if $t < 1$,
$i = 100e^{-0.1(t-1)}$ if $1 < t < 1.01$,
$i = 100[e^{-0.1(t-1)} - e^{-0.1(t-1.01)}] = -0.1106e^{-0.1t}$ if $t > 1.01$

38. $i = 0$ if $t < 3$, $\quad i = 5 - 5e^{-0.1(t-3)} = 5(1 - 1.3499e^{-0.1t})$ if $t > 3$

40. CAS PROJECT. Students should become aware of the fact that careful observation of plots may lead to discoveries or to more information about conjectures that they may want to prove or disprove. The curves branch from the solution of the homogeneous equation at the instant at which the impulse is applied, which by choosing, say, $a = 1, 2, 3, \cdots$, gives an interesting joint plot.

SECTION 5.4. Differentiation and Integration of Transforms, page 275

Purpose. To show that, ***roughly,*** differentiation and integration of transforms (not of functions, as before!) corresponds to multiplication and division, respectively, of functions by t, with application to the derivation of further transforms and to the solution of Laguerre's differential equation.

Comment on Application to Variable-Coefficient Equations

This possibility is rather limited; our Example 4 is perhaps the best elementary example of practical interest.

Very Short Courses. This section and the two subsequent sections can be omitted.

SOLUTIONS TO PROBLEM SET 5.4, page 278

2. $-\left(\dfrac{12}{s^2-16}\right)' = \dfrac{24s}{(s^2-16)^2}$

4. $-\left(\dfrac{s+1}{(s+1)^2+1}\right)' = -\dfrac{s^2+2s+2-(s+1)(2s+2)}{(s^2+2s+2)^2} = \dfrac{s^2+2s}{(s^2+2s+2)^2}$

6. $\dfrac{d^2}{ds^2}\left(\dfrac{2}{(s^2+4)}\right) = \dfrac{d}{ds}\left(\dfrac{-4s}{(s^2+4)^2}\right) = \dfrac{-4(s^2+4)^2+16s^2(s^2+4)}{(s^2+4)^4}$

$$= \frac{12s^2-16}{(s^2+4)^3}$$

8. $\dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2+\omega^2}\right) = \dfrac{d}{ds}\dfrac{-s^2+\omega^2}{(s^2+\omega^2)^2} = \dfrac{2s^3-6\omega^2 s}{(s^2+\omega^2)^3}$

10. $\left(\dfrac{3}{s^2-9}\right)' = -\dfrac{6s}{(s^2-9)^2}$. *Answer:* $\frac{1}{6}t\sinh 3t$

12. By (6),

$$\int_s^\infty \frac{2\tilde{s}+6}{(\tilde{s}^2+6\tilde{s}+10)^2}\,d\tilde{s} = \frac{1}{s^2+6s+10} = \mathcal{L}\left(\frac{f}{t}\right).$$

The inverse transform of the integral is $e^{-3t}\sin t$. *Answer:*

$$te^{-3t}\sin t.$$

14. $[\ln(s+a)-\ln(s+b)]' = \dfrac{1}{s+a} - \dfrac{1}{s+b}$ has the inverse transform $e^{-at}-e^{-bt}$, so that (1) gives the *answer*

$$\frac{e^{-bt}-e^{-at}}{t}.$$

16. We have

$$\operatorname{arc\,cot}\frac{s}{\pi} = \int_s^\infty \frac{\pi}{\tilde{s}^2+\pi^2}\,d\tilde{s}.$$

The inverse transform of the integrand is $\sin \pi t$. From (6) we thus obtain the *answer*

$$t^{-1}\sin \pi t.$$

18. $n!/(s-a)^{n+1}$

20. CAS PROJECT. Students should become aware of the fact that usually there are various possibilities for calculations, and they should not rush into numerical work before making a careful selection of formulas.

(b) The formula follows by the usual rule of differentiating a product n times. Some of the polynomials are

$$l_2 = 1 - 2t + \tfrac{1}{2}t^2$$
$$l_3 = 1 - 3t + \tfrac{3}{2}t^2 - \tfrac{1}{6}t^3$$
$$l_4 = 1 - 4t + 3t^2 - \tfrac{2}{3}t^3 + \tfrac{1}{24}t^4$$
$$l_5 = 1 - 5t + 5t^2 - \tfrac{5}{3}t^3 + \tfrac{5}{24}t^4 - \tfrac{1}{120}t^5.$$

SECTION 5.5. Convolution. Integral Equations, page 279

Purpose. To find the inverse $h(t)$ of a product $H(s) = F(s)G(s)$ of transforms whose inverses are known.

Main Content, Important Concepts

Convolution $f * g$, its properties

Convolution theorem

Application to differential and integral equations

Comment on Occurrence

In a differential equation, the transform $R(s)$ of the right side $r(t)$ is known from Step 1. By solving the subsidiary equation algebraically for $Y(s)$ the transform $R(s)$ gets multiplied by the reciprocal of the factor of $Y(s)$ on the left (the transfer function $Q(s)$; see Sec. 5.2). This calls for the convolution theorem, unless one sees some other way or shortcut.

Very Short Courses. This section can be omitted.

SOLUTIONS TO PROBLEM SET 5.5, page 283

2. $1 * \sin \omega t = \int_0^t \sin \omega\tau \, d\tau = -\frac{\cos \omega\tau}{\omega}\Big|_0^t = \frac{1 - \cos \omega t}{\omega}$

4. This is similar to Example 1. We obtain

$$\int_0^t \cos \omega\tau \cos (\omega t - \omega\tau) \, d\tau = \frac{1}{2} \int_0^t [\cos \omega t + \cos (2\omega\tau - \omega t)] \, d\omega$$

$$= \frac{1}{2}\left[t \cos \omega t + \frac{\sin \omega t - \sin(-\omega t)}{2\omega}\right] = \frac{1}{2} t \cos \omega t + \frac{1}{2\omega} \sin \omega t.$$

6. $e^{at} * e^{bt} = \int_0^t e^{a\tau} e^{b(t-\tau)} \, d\tau = e^{bt} \int_0^t e^{(a-b)\tau} \, d\tau = \frac{e^{at} - e^{bt}}{a - b}$

8. $\int_0^t (t - \tau)^2 u(\tau - 1) \, d\tau = \int_1^t (t - \tau)^2 \, d\tau = -\frac{1}{3}(t - \tau)\Big|_1^t = \frac{1}{3}(t - 1)^3$ if $t > 1$ and 0 if $t < 1$.

10. $6 * e^{-3t} = \int_0^t 6e^{-3\tau} \, d\tau = -2e^{-3t} + 2$

12. $e^{at} * e^{at} = \int_0^t a^{a\tau} e^{a(t-\tau)} \, d\tau = e^{at} \int_0^t d\tau = te^{at}$

14. $\cos \omega t * \cos \omega t$. Proceed as in Prob. 4. *Answer:*

$$\frac{1}{2} t \cos \omega t + \frac{1}{2\omega} \sin \omega t$$

16. $u(t - a) * e^{2t} = \int_a^t e^{2(t-\tau)} \, d\tau = e^{2t} \int_a^t e^{-2\tau} \, d\tau = \frac{1}{2}(e^{2(t-a)} - 1)$ if $t > a$ and 0 if $t < a$

18. $e^{-3t} * e^{2t} = \int_0^t e^{-3\tau}e^{2(t-\tau)}\,d\tau = e^{2t}\int_0^t e^{-5\tau}\,d\tau$

$$= \frac{e^{2t}}{5}(1 - e^{-5t}) = \frac{1}{5}(e^{2t} - e^{-3t})$$

20. Subsidiary equation $s^2Y + Y = s^{-2}$, $Y = 1/(s^2 + s^4)$, solution $y = t - \sin t$

22. The subsidiary equation is

$$(s^2 + 3s + 2)Y = 1 + (1 - e^{-s})/s.$$

From this,

$$Y = \frac{s + 1 - e^{-s}}{s(s^2 + 3s + 2)}.$$

Answer:

$$y = \tfrac{1}{2}(1 - e^{-2t}) + \tfrac{1}{2}(2e^{-(t-1)} - e^{-2(t-1)} - 1)u(t - 1).$$

24. We use the notation of the text,

$$q = e^{3t} - e^{2t}, \qquad \mathcal{L}^{-1}\{[(s + a)y(0) + y'(0)]Q\} = -4e^{3t} + 5e^{2t}.$$

Then

$$\begin{aligned} r * q &= 2e^{3t} - 4e^{2t} + 2e^{t} && \text{if } 0 < t < 2 \\ y &= -2e^{3t} + e^{2t} + 2e^{t} && \text{if } 0 < t < 2 \\ r * q &= 2(1 - e^{-4})e^{3t} + 4(e^{-2} - 1)e^{2t} && \text{if } t > 2 \\ y &= -(2 + 2e^{-4})e^{3t} + (1 + 4e^{-2})e^{2t} && \text{if } t > 2. \end{aligned}$$

26. We use the notation of the text,

$$q = e^{-t} - e^{-2t}, \; r * q = 4e^{-t} - e^{-2t} + 2t - 3 \qquad \text{if } 0 < t < 1.$$

For $t > 1$ we have

$$\begin{aligned} r * q &= \int_0^1 \tau q(t - \tau)\,d\tau + \int_1^t 8q(t - \tau)\,d\tau \\ &= \left[4e^{-t} - (1 + e^2)e^{-2t}\right] + \left[4 - 8e^{-(t-1)} + 4e^{-2(t-1)}\right] \\ &= (4 - 8e)e^{-t} + (3e^2 - 1)e^{-2t} + 4. \end{aligned}$$

28. $Y = 2s^{-2} - 4s^{-2}Y$, $Y = 2/(s^2 + 4)$, $y = \sin 2t$

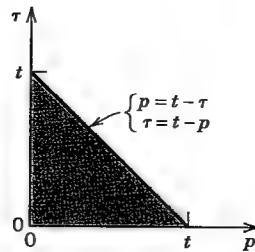
30. $Y = 2/(s^2 + 4) + 2Y/(s^2 + 4)$, $Y = 2/(s^2 + 2)$, $y = \sqrt{2}\sin\sqrt{2}t$

32. $Y = 1/(s^2 + 1) + Y/(s^2 + 1)$, $Y = 1/s^2$, $y = t$

34. TEAM PROJECT. (a) Setting $t - \tau = p$, we have $\tau = t - p$, $d\tau = -dp$, and p runs from t to 0; thus

$$\begin{aligned} f * g &= \int_0^t f(\tau)g(t - \tau)\,d\tau = \int_t^0 g(p)f(t - p)(-dp) \\ &= \int_0^t g(p)f(t - p)\,dp = g * f. \end{aligned}$$

(b) Interchanging the order of integration and noting that we integrate over the shaded triangle in the figure we obtain

$$
\begin{aligned}
(f * g) * v &= v * (f * g) \\
&= \int_0^t v(p) \int_0^{t-p} f(\tau) g(t - p - \tau)\, d\tau\, dp \\
&= \int_0^t f(\tau) \int_0^{t-\tau} g(t - \tau - p) v(p)\, dp\, d\tau \\
&= f * (g * v).
\end{aligned}
$$

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(c) This is a simple consequence of the additivity of the integral.

(d) $\mathcal{L}(\delta * f) = \mathcal{L}(\delta)\mathcal{L}(f) = 1 \cdot \mathcal{L}(f) = \mathcal{L}(f)$, since $\mathcal{L}\{\delta(t)\} = 1$ by (8), Sec. 5.3.

(e) Let $t > k$. Then $(f_k * f)(t) = \int_0^k \frac{1}{k} f(t - \tau)\, d\tau = f(t - \tilde{t})$ for some $\tilde{t}$ between 0 and k. Now let $k \to 0$. Then $\tilde{t} \to 0$ and $f_k(t - \tilde{t}) \to \delta(t)$, so that the formula follows.

(f) $s^2Y - sy(0) - y'(0) + \omega^2 Y = \mathcal{L}(r)$ has the solution

$$Y = \frac{1}{\omega}\left(\frac{\omega}{s^2 + \omega^2}\right)\mathcal{L}(r) + y(0)\frac{s}{s^2 + \omega^2} + \frac{y'(0)}{\omega}\,\frac{\omega}{s^2 + \omega^2}$$

etc.

SECTION 5.6. Partial Fractions. Differential Equations, page 284

Purpose. This section is mainly for reference. Partial fractions are discussed systematically in terms of examples, along with their inverse transforms.

Very Short Courses. Omit this section.

SOLUTIONS TO PROBLEM SET 5.6, page 289

2. $t^2 + \sin t$ **4.** $(1 - t)e^{-t}$ **6.** $e^t(\cos t + t \sin t)$ **8.** $te^t - \frac{1}{2}t^2 e^{2t}$

14. The subsidiary equation is

$$s^2 Y + \omega_0^2 Y = K \frac{p}{s^2 + p^2}.$$

Its solution is

$$Y(s) = \frac{Kp}{(s^2 + \omega_0^2)(s^2 + p^2)} \qquad (\omega_0^2 \neq p^2).$$

Since $\omega_0^2 \neq p^2$ by assumption, $s^2 + \omega_0^2$ is the product of two unrepeated complex factors, and so is $s^2 + p^2$. Accordingly, the partial fraction representation is

$$Y(s) = \frac{Kp}{(s^2 + \omega_0^2)(s^2 + p^2)} = \frac{As + B}{s^2 + \omega_0^2} + \frac{Ms + N}{s^2 + p^2}.$$

Multiplication by the common denominator gives

$$Kp = (s^2 + p^2)(As + B) + (s^2 + \omega_0^2)(Ms + N).$$

Equating the coefficients of each power of s on both sides gives the four equations
(a) $[s^3]$: $\quad 0 = A + M$, thus $M = -A$
(b) $[s^2]$: $\quad 0 = B + N$, thus $N = -B$
(c) $[s]$: $\quad 0 = p^2A + \omega_0^2M = (p^2 - \omega_0^2)A$ by (a); hence $A = M = 0$
(d) $[s^0]$: $Kp = p^2B + \omega_0^2N = (p^2 - \omega_0^2)B$ by (b); hence $B = Kp/(p^2 - \omega_0^2)$.
From this, with $N = -B$, we have

$$Y(s) = \frac{Kp}{p^2 - \omega_0^2}\left(\frac{1}{s^2 + \omega_0^2} - \frac{1}{s^2 + p^2}\right).$$

The inverse is (see Table 5.1 in Sec. 5.1)

$$y(t) = \frac{Kp}{p^2 - \omega_0^2}\left(\frac{1}{\omega_0}\sin \omega_0 t - \frac{1}{p}\sin pt\right).$$

This is a superposition of two harmonic oscillations, as expected.

16. TEAM PROJECT. (a) If $f(t)$ is piecewise continuous on an interval of length p, then its Laplace transform exists, and we can write the integral from zero to infinity as the series of integrals over successive periods:

$$\mathcal{L}(f) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^p e^{-st}f\,dt + \int_p^{2p} e^{-st}f\,dt + \int_{2p}^{3p} e^{-st}f\,dt + \cdots.$$

If we substitute $t = \tau + p$ in the second integral, $t = \tau + 2p$ in the third integral, $\cdots, t = \tau + (n-1)p$ in the nth integral, $\cdots$, then the new limits in every integral are 0 and p. Since

$$f(\tau + p) = f(\tau), \qquad f(\tau + 2p) = f(\tau),$$

etc., we thus obtain

$$\mathcal{L}(f) = \int_0^p e^{-s\tau}f(\tau)\,d\tau + \int_0^p e^{-s(\tau+p)}f(\tau)\,d\tau + \int_0^p e^{-s(\tau+2p)}f(\tau)\,d\tau + \cdots.$$

The factors that do not depend on τ can be taken out from under the integral signs; this gives

$$\mathcal{L}(f) = [1 + e^{-sp} + e^{-2sp} + \cdots]\int_0^p e^{-s\tau}f(\tau)\,d\tau.$$

The series in brackets $[\cdots]$ is a geometric series whose sum is $1/(1 - e^{-ps})$. The theorem now follows.

(b) From (10) we obtain

$$\mathcal{L}(f) = \frac{1}{1 - e^{-2\pi s/\omega}}\int_0^{\pi/\omega} e^{-st}\sin \omega t\,dt.$$

Using $1 - e^{-2\pi s/\omega} = (1 + e^{-\pi s/\omega})(1 - e^{-\pi s/\omega})$ and integrating by parts or noting that the integral is the imaginary part of the integral

$$\int_0^{\pi/\omega} e^{(-s+i\omega)t}\,dt = \frac{1}{-s+i\omega} e^{(-s+i\omega)t}\Big|_0^{\pi/\omega} = \frac{-s-i\omega}{s^2+\omega^2}(-e^{-s\pi/\omega} - 1)$$

we obtain the result.

(c) From (10) we obtain the following equation by using $\sin \omega t$ from 0 to π/ω and $-\sin \omega t$ from π/ω to $2\pi/\omega$:

$$\frac{\omega}{s^2+\omega^2}\,\frac{1+e^{\pi s/\omega}}{e^{\pi s/\omega}-1} = \frac{\omega}{s^2+\omega^2}\,\frac{e^{-\pi s/2\omega}+e^{\pi s/2\omega}}{e^{\pi s/2\omega}-e^{-\pi s/2\omega}}$$

$$= \frac{\omega}{s^2+\omega^2}\,\frac{\cosh(\pi s/2\omega)}{\sinh(\pi s/2\omega)}.$$

This gives the result.

(d) The saw-tooth wave has the representation

$$f(t) = \frac{k}{p}t \qquad \text{if } 0 < t < p, \qquad f(t+p) = f(t).$$

Integration by parts gives

$$\int_0^p e^{-st}t\,dt = -\frac{t}{s}e^{-st}\Big|_0^p + \frac{1}{s}\int_0^p e^{-st}\,dt$$

$$= -\frac{p}{s}e^{-sp} - \frac{1}{s^2}(e^{-sp} - 1),$$

and thus from (10) we obtain the result

$$\mathcal{L}(f) = \frac{k}{ps^2} - \frac{ke^{-ps}}{s(1-e^{-ps})} \qquad (s > 0).$$

(e) Since kt/p has the transform k/ps^2, from (d) we have the result

$$\frac{ke^{-ps}}{s(1-e^{-ps})} \qquad (s > 0).$$

SECTION 5.7. Systems of Differential Equations, page 291

Purpose. This new section explains the application of the Laplace transform to systems of differential equations in terms of three typical examples: a mixing problem, an electrical network, and a system of masses on elastic springs.

SOLUTIONS TO PROBLEM SET 5.7, page 294

2. The subsidiary equations

$$sY_1 + 3 = 6Y_1 + 9Y_2, \qquad sY_2 + 3 = Y_1 + 6Y_2$$

give

$$Y_1 = -\frac{3s+9}{(s-9)(s-3)} = -\frac{6}{s-9} + \frac{3}{s-3}, \qquad y_1 = -6e^{9t} + 3e^{3t}$$

$$Y_2 = -\frac{3s-15}{(s-9)(s-3)} = -\frac{2}{s-9} - \frac{1}{s-3}, \qquad y_2 = -2e^{9t} - e^{3t}.$$

4. $sY_1 + 3 = 5Y_1 + Y_2, \quad sY_2 - 7 = Y_1 + 5Y_2.$ *Answer:*

$$y_1 = 2e^{6t} - 5e^{4t}, \qquad y_2 = 2e^{6t} + 5e^{4t}.$$

6. $y_1 = \sin t + \cos 2t, \quad y_2 = \sin t - \cos 2t$

8. The subsidiary equations are

$$s^2Y_1 - 3s = -5Y_1 + 2Y_2, \qquad s^2Y_2 - s = 2Y_1 - 2Y_2.$$

They have the solutions

$$Y_1 = \frac{3s^3 + 8s}{(s^2+1)(s^2+6)} = \frac{s}{s^2+1} + \frac{2s}{s^2+6}$$

$$Y_2 = \frac{s^3 + 11s}{(s^2+1)(s^2+6)} = \frac{2s}{s^2+1} - \frac{s}{s^2+6}.$$

Answer:

$$y_1 = \cos t + 2\cos\sqrt{6}\,t, \qquad y_2 = 2\cos t - \cos\sqrt{6}\,t.$$

10. $y_1 = t^2, y_2 = t^2 + 2t, y_3 = t^2 - 2t$

12. The subsidiary equations are

$$sY_1 + Y_2 = \frac{2s}{s^2+1}(1 - e^{-2\pi s}), \qquad Y_1 + sY_2 = 1.$$

Solving algebraically gives

$$Y_1 = \frac{1}{s^2+1} - \frac{2s^2e^{-2\pi s}}{s^4-1}$$

$$Y_2 = \frac{s}{s^2+1} + \frac{2se^{-2\pi s}}{s^4-1}.$$

Answer:

$$y_1 = \sin t \quad \text{if } 0 \leqq t \leqq 2\pi, \qquad y_1 = -\sinh(t - 2\pi) \quad \text{if } t > 2\pi$$

$$y_2 = \cos t \quad \text{if } 0 \leqq t \leqq 2\pi, \qquad y_2 = \cosh(t - 2\pi) \quad \text{if } t > 2\pi.$$

14. The subsidiary equations are

$$sY_1 + 4 = 64\left(\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}\right) + 2Y_1 + 4Y_2, \qquad sY_2 + 4 = Y_1 + 2Y_2.$$

Solving algebraically gives

$$Y_1 = \frac{-4s - 8}{s(s-4)} - \frac{64(2 + s - s^2)e^{-s}}{s^3(s-4)}$$

$$Y_2 = \frac{-4s + 4}{s(s-4)} + \frac{64(1+s)e^{-s}}{s^3(s-4)}.$$

Taking the inverse Laplace transform gives the *answer*

$$y_1 = -6e^{4t} + 2 + u(t-1)[-18 + 10e^{4t-4} - 8t + 16t^2]$$

$$y_2 = -3e^{4t} - 1 + u(t-1)[7 + 5e^{4t-4} - 4t - 8t^2].$$

16. $y_1 = 100 - 62.5e^{-0.24t} - 37.5e^{-0.08t}$,
$y_2 = 100 + 125e^{-0.24t} - 75e^{-0.08t}$.
Setting $2t = \tau$ gives the old solution, except for notation.

18. $y_1 = \cos \sqrt{3}t + \frac{2}{3} \sin 3t + \sin t$
$y_2 = \cos \sqrt{3}t - \frac{2}{3} \sin 3t - \sin t$

20. For $0 \leqq t \leqq 2\pi$ the solution is as in Prob. 19,

$$i_1 = 2e^{-8t} + 13e^{-2t} - 15 \cos t + 42 \sin t.$$

For $t > 2\pi$ one has to add to this further terms whose form is determined by this solution and the second shifting theorem,

$$u(t - 2\pi)[-2e^{-8(t-2\pi)} - 13e^{-2(t-2\pi)} + 15 \cos t - 42 \sin t].$$

The cosine and sine terms cancel, so that

$$i_1 = 2(1 - e^{16\pi})e^{-8t} + 13(1 - e^{4\pi})e^{-2t} \quad \text{if} \quad t > 2\pi.$$

Similarly, for i_2 we obtain

$$i_2 = \begin{cases} -e^{-8t} + 13e^{-2t} - 12 \cos t + 18 \sin t & \text{if} \quad 0 \leqq t \leqq 2\pi \\ -(1 - e^{16\pi})e^{-8t} + 13(1 - e^{4\pi})e^{-2t} & \text{if} \quad t > 2\pi. \end{cases}$$

SOLUTIONS TO CHAPTER 5 REVIEW, page 299

16. $\dfrac{\pi}{(s+1)^2 + \pi^2}$

18. $\dfrac{2\pi^2}{4s(s^2 + \pi^2)}$

20. $e^{-s/4} \dfrac{s^2 + 8s + 32}{16s^3}$

22. $\dfrac{s}{(s-2)(s^2+16)}$

24. $\dfrac{2s^2}{(s^2+1)^2}$

26. $\dfrac{1}{6}(e^{4t} - e^{-2t})$

28. $3t^2 + t^3$

30. $e^{3t} + \cos 2t$

32. $\sin(\omega t + \theta)$

34. $y = \begin{cases} 2.5 \cos t & \text{if} \quad 0 < t < 2 \\ 2.5 \cos t + \sin(t-2) & \text{if} \quad t > 2 \end{cases}$

36. $e^{-t}(\cos t - \sin t) + e^t(15 \cos t - 29 \sin t)$

38. 0 if $0 \leqq t \leqq 2$ and $1 - 2e^{-(t-2)} + e^{-2(t-2)}$ if $t > 2$

40. $y_1 = -6e^{4t} + 2, \quad y_2 = -3e^{4t} - 1$

42. $y_1 = 3e^{2t} + e^{-5t}, \quad y_2 = 4e^{2t} - e^{-5t}$

44. $y_1 = (1/\sqrt{10}) \sin \sqrt{10}\, t, \quad y_2 = -(1/\sqrt{10}) \sin \sqrt{10}\, t$

46. $q = \begin{cases} 1 - \frac{1}{2}(e^{-t} + \cos t + \sin t) & \text{if} \quad 0 < t < \pi \\ \frac{1}{2}[(e^{-\pi} - 3) \cos t - (e^{-\pi} + 1) \sin t] & \text{if} \quad t > \pi, \end{cases}$

$i(t) = q'(t)$

48. $i_1 = 2(1 - e^{-t}), \quad i_2 = 2e^{-t}$

50. $5i_1' + 20(i_1 - i_2) = 60, \quad 30i_2' + 20(i_2' - i_1') + 20i_2 = 0.$ *Answer:*

$$i_1 = -8e^{-2t} + 5e^{-0.8t} + 3, \qquad i_2 = -4e^{-2t} + 4e^{-0.8t}$$

PART B. LINEAR ALGEBRA, VECTOR CALCULUS

Major Change

Part B consists of

Chap. 6 Linear Algebra: Matrices, Vectors, Determinants. Linear Systems of Equations

Chap. 7 Linear Algebra: Matrix Eigenvalue Problems

Chap. 8 Vector Differential Calculus. Grad, Div, Curl

Chap. 9 Vector Integral Calculus. Integral Theorems

Following several requests, we now present eigenvalue problems in a separate chapter. However, this does not change the flow of the material in Part B as a whole.

Chapter 8 is self-contained and completely independent of Chaps. 6 and 7. Thus, Part B consists of two large **independent** units, namely Linear Algebra (Chaps. 6, 7) and Vector Calculus (Chaps. 8, 9). Chapter 9 depends on Chap. 8, mainly because of the occurrence of div and curl (defined in Chap. 8) in the Gauss and Stokes theorems in Chap. 9.

CHAPTER 6 Linear Algebra: Matrices, Vectors, Determinants. Linear Systems of Equations

Major Changes

Various local changes have been made in order to increase the usefulness of this chapter for applications. By cutting out some passages that were somewhat sluggish and less important in practice, the total amount of material has been reduced slightly, resulting in a smoother and better motivated flow of ideas and methods and a corresponding valuable gain in teaching time. More specifically, there are essentially three major changes, as follows.

1. The beginning, which had been somewhat slow by modern standards, has been streamlined, so that the student will see **applications** to linear systems of equations much earlier.
2. The reference section on second-order and third-order determinants, which had become somewhat dated, has been omitted and replaced by a shorter portion on that material at the beginning of the section on determinants (Sec. 6.6), from which the essential information on those lower order determinants can now be obtained more easily and quickly.
3. The two sections on determinants and Cramer's rule have been combined into a single section (Sec. 6.6), which precedes the discussion of the inverse in Sec. 6.7—thus making this portion of the chapter more compact.

SECTION 6.1. Basic Concepts. Matrix Addition, Scalar Multiplication, page 305

Purpose. Explanation of the basic concepts and the two basic matrix operations.

Main Content, Important Concepts

Matrix, square matrix, main diagonal

Double subscript notation

Row vector, column vector, transposition

Equality of matrices

Matrix addition

Scalar multiplication (multiplication of a matrix by a scalar)

Comment on Notation

For transposition, T seems preferable over a prime, which is often used in the literature, but will be needed to indicate differentiation in Chap. 8.

Comments on Important Facts

One should emphasize that vectors are always included as special cases of matrices and that those two operations have properties [formulas (4), (5)] similar to those of operations for numbers, which is a great practical advantage.

Comment on Vector Spaces

Since vector spaces are defined in terms of matrix addition and scalar multiplication, they could be mentioned here. We discuss them later, in Sec. 6.4, when the student will be more familiar with the matrix concept.

SOLUTIONS TO PROBLEM SET 6.1, page 309

2. $\begin{bmatrix} 24 & -36 \\ 4 & -36 \end{bmatrix}$, $\begin{bmatrix} -24 & 36 \\ -4 & 36 \end{bmatrix}$, $\begin{bmatrix} -24 & 36 \\ -4 & 36 \end{bmatrix}$

4. $\mathbf{C}$, $\begin{bmatrix} 26 & -3 \\ 0 & 8 \\ 1 & 3 \end{bmatrix}$, undefined (not of the same size)

6. Undefined, undefined, the 2×2 zero matrix $\mathbf{0}$

8. $\begin{bmatrix} 108 & 0 & -54 \\ -48 & 72 & 132 \end{bmatrix}$, the same matrix because of (5), (6), and $(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}$.

10. $[-6 \quad -5 \quad -3]$, $[6 \quad 5 \quad 3]^{\mathsf{T}}$, undefined

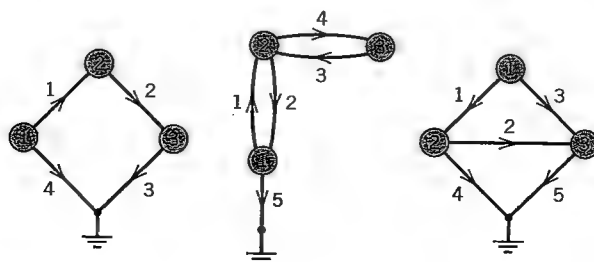
12. $[11 \quad -1 \quad 3]$, $[-36 \quad 120 \quad -48]$

14. Undefined, $[6 \quad 5 \quad 3]^{\mathsf{T}}$, undefined (not of the same size)

16. $\mathbf{0}$ by (7), undefined, $[7 \quad 0 \quad 26]$

20. TEAM PROJECT. (b) The nodal incidence matrices are

$$\begin{bmatrix} -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

(c) The networks with these incidence matrices are

SECTION 6.2. Matrix Multiplication, page 311

Purpose. Matrix multiplication, the third and last algebraic operation, is defined and discussed, with emphasis on its "unusual" properties; this also includes its representation by inner products of row and column vectors.

Main Content, Important Facts

Definition of matrix multiplication ("rows times columns")

Properties of matrix multiplication

Matrix products in terms of inner products of vectors

Linear transformations motivating the definition of multiplication

$\mathbf{AB} \neq \mathbf{BA}$ in general, so the order of factors is important.

$\mathbf{AB} = \mathbf{0}$ does not imply $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or $\mathbf{BA} = \mathbf{0}$.

$(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$

Short Courses. Products in terms of row and column vectors and the discussion of linear transformations could be omitted.

Comments on Content

Most important for the next sections on systems of equations will be the multiplication of a matrix times a vector.

"Unusual properties" (i.e., having no counterpart in the multiplication of numbers) are exhibited in Examples 4 and 5, and it may be good to invite the student to invent further examples. The student should also get used to cases in which products are not defined, in order to recognize the limitation of the definition.

Formula (5) for the transposition of a product should be memorized.

In motivating matrix multiplication by linear transformations, one may also illustrate the geometric significance of noncommutativity by combining a rotation with a stretch in x-direction in both orders and show that a circle transforms into an ellipse with main axes in the direction of the coordinate axes or rotated, respectively.

SOLUTIONS TO PROBLEM SET 6.2, page 319

2. $\begin{bmatrix} 34 \\ 15 \\ 11 \end{bmatrix}$, $\begin{bmatrix} 248 \\ 237 \\ 102 \end{bmatrix}$, $\begin{bmatrix} 2618 \\ 1794 \\ 1105 \end{bmatrix}$

4. $[34 \quad 15 \quad 11]^T$, undefined, $[34 \quad 24 \quad 17]$

6. $\begin{bmatrix} 5 & 20 & 15 \\ 20 & 80 & 60 \\ 15 & 60 & 45 \end{bmatrix}$, 130, $[-6 \quad 14]$

8. -32, -32 [as follows from the first result and (5)], $\begin{bmatrix} 8 & 2 \\ 12 & -15 \\ 4 & -1 \end{bmatrix}$, and the transpose of it [again by (5)]

10. TEAM PROJECT. **(a)** b_{jk} is the dot product of the jth row of **A** and the kth column of $\mathbf{A}^T$, which is the kth *row* of **A** because of the transposition. Thus,

$$b_{jm} = \sum_m a_{jm}a_{km} = \sum_m a_{km}a_{jm} = b_{kj}.$$

To prove the second statement in (a), use (5). If $\mathbf{AB} = \mathbf{BA}$, then

$$(\mathbf{AB})^T = (\mathbf{BA})^T = \mathbf{A}^T\mathbf{B}^T = \mathbf{AB}$$

because $\mathbf{A}^T = \mathbf{A}$, $\mathbf{B}^T = \mathbf{B}$, by assumption of symmetry. Conversely, if $(\mathbf{AB})^T = \mathbf{AB}$, then $\mathbf{AB} = (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{BA}$, so that **A** and **B** commute.

(b) Idempotent are $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, etc.; nilpotent are $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$, etc., and $\mathbf{A}^2 = \mathbf{I}$ is true for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ a & -1 \end{bmatrix}, \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & c \\ 1/c & 0 \end{bmatrix}$$

where a, b, and $c \neq 0$ are arbitrary.

(c) Triangular are $\mathbf{U}_1 + \mathbf{U}_2$, $\mathbf{U}_1\mathbf{U}_2$, hence $\mathbf{U}_1^2$ and the corresponding expressions for $\mathbf{L}_1$ and $\mathbf{L}_2$. $\mathbf{U}_1^T$ is lower triangular.

(d) The entry c_{kj} of $(\mathbf{AB})^T$ is c_{jk} of **AB**, which is row j of **A** times column k of **B**. On the right, c_{kj} is row k of $\mathbf{B}^T$, hence column k of **B**, times column j of $\mathbf{A}^T$, hence row j of **A**.

12. The transition probabilities can be given in a matrix

$$\begin{matrix} & \text{To } N & \text{To } T & \\ \mathbf{A} = & \left[\begin{matrix} 0.8 \\ 0.5 \end{matrix}\right. & \left.\begin{matrix} 0.2 \\ 0.5 \end{matrix}\right] & \begin{matrix} \text{From } N \\ \text{From } T \end{matrix} \end{matrix}$$

The first row gives the state after one day if initially there was N, and the second row if initially there was T. From this we see that there will be N after 2 days with probability

$$0.8 \cdot 0.8 + 0.2 \cdot 0.5 = 0.74$$

because N will remain with $P = 0.8$ and T will return to N with $P = 0.5$. Similarly for the other possibilities. We see that this is just the law of matrix multiplication. Accordingly, $\mathbf{A}^2$ gives the probabilities after 2 days and $\mathbf{A}^3$ after 3 days; here, by calculation,

$$\mathbf{A}^2 = \begin{bmatrix} 0.74 & 0.26 \\ 0.65 & 0.35 \end{bmatrix}, \qquad \mathbf{A}^3 = \begin{bmatrix} 0.722 & 0.278 \\ 0.695 & 0.305 \end{bmatrix}.$$

Answer: 0.26, 0.278.

14. The matrix of the transition probabilities is

$$\mathbf{A} = \begin{bmatrix} 0.9 & 0.1 \\ 0.002 & 0.998 \end{bmatrix}.$$

The starting vector is $\mathbf{x}_0 = [1200 \quad 98800]$ and gives (rounded)

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_0\mathbf{A} = [1278 \quad 98722] \\ \mathbf{x}_2 &= \mathbf{x}_1\mathbf{A} = [1347 \quad 98653] \\ \mathbf{x}_3 &= \mathbf{x}_2\mathbf{A} = [1410 \quad 98590] \end{aligned}$$

indicating that a substantial increase is likely.

16. We then proceed by time intervals of 10 years.

18. TEAM PROJECT. (b) Use induction on n. True if $n = 1$. Take the formula in the problem as the induction hypothesis, multiply by $\mathbf{A}$, and simplify the entries in the product by the addition formulas for the cosine and sine to get $\mathbf{A}^{n+1}$.

(c) Those formulas follow directly from the definition of matrix multiplication.

(d) A scalar matrix would correspond to a stretch or contraction by the same factor in all directions.

(e) Rotations about the x_1-, x_2-, x_3-axes through θ, φ, ψ, respectively.

SECTION 6.3. Linear Systems of Equations. Gauss Elimination, page 321

Purpose. This simple section centers around the Gauss elimination for solving linear systems of m equations in n unknowns $x_1, \cdots, x_n$, its practical use as well as its mathematical justification (leaving the—more demanding—general existence theory to the next sections).

Main Content, Important Concepts

Nonhomogeneous, homogeneous, coefficient matrix, augmented matrix

Gauss elimination in the case of the existence of

I. a unique solution (Examples 2, 4)

II. infinitely many solutions (Example 3)

III. no solutions (Example 5).

Pivoting

Elementary row operations, echelon form

Background Material. All one needs here is the multiplication of a matrix and a vector.

Comments on Content

The student should become aware of the following facts:

1. Linear systems of equations provide a major application of matrix algebra and justification of the definitions of its concepts.

2. The Gauss elimination (with pivoting) gives sensible results in each of the Cases I–III.

3. This method is a *systematic* elimination that does not look for unsystematic "shortcuts" (depending on the size of the numbers involved and still advocated in some older pre-computer-age books).

Algorithms for programs of Gauss's and related methods are discussed in Sec. 18.1, which is independent of the rest of Chap. 18, and can thus be taken up along with the present section in case of time and interest.

SOLUTIONS TO PROBLEM SET 6.3, page 329

2. $x = 1, y = -2$

4. $x = 2, y = 0, z = -4$

6. No solution

8. $x = 2y + 1, z = 4$

10. $x = 7y - 9z$

12. No solution

14. $w = x - 2y, z = 3$

16. $w = 0, x = 3z, y = 2z + 1$

18. Currents at the lower node:

$$-I_1 + I_2 + I_3 = 0$$

(minus because I_1 flows out). Voltage in the left circuit:

$$4I_1 + 12I_2 = 12 + 24$$

and in the right circuit

$$12I_2 - 8I_3 = 24$$

(minus because I_3 flows against the arrow of E_2). Hence the augmented matrix of the system is

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 4 & 12 & 0 & 36 \\ 0 & 12 & -8 & 24 \end{bmatrix}.$$

The solution is

$$I_1 = \tfrac{27}{11}, \quad I_2 = \tfrac{24}{11}, \quad I_3 = \tfrac{3}{11} \text{ ampere.}$$

22. $P_1 = 6, P_2 = 10, D_1 = S_1 = 18, D_2 = S_2 = 26$

24. PROJECT. (a) **B** and **C** are different. For instance, it makes a difference whether we first multiply a row and then interchange, and then do these operations in reverse order.

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \\ a_{21} - 5a_{11} & a_{22} - 5a_{12} \\ 8a_{41} & 8a_{42} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} - 5a_{11} & a_{32} - 5a_{12} \\ a_{21} & a_{22} \\ 8a_{41} & 8a_{42} \end{bmatrix}$$

(b) Premultiplying **A** by **E** makes **E** operate on *rows* of **A**. The assertions then follow almost immediately from the definition of matrix multiplication.

(c) These matrices, applied in the order $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, are

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix}$$

with the multipliers given by

$$m_{21} = \frac{a_{21}}{a_{11}}, \qquad m_{31} = \frac{a_{31}}{a_{11}}, \qquad m_{32} = \frac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}a_{22} - a_{12}a_{21}}.$$

The product is

$$\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{bmatrix}.$$

SECTION 6.4. Rank of a Matrix. Linear Independence. Vector Space, page 331

Purpose. This section introduces some theory centered around linear independence and rank, in preparation for the discussion of the existence and uniqueness problem for linear systems of equations (Sec. 6.5).

Main Content, Important Concepts

Linear independence

Real vector space R^n, dimension, basis

Rank defined in terms of row vectors

Rank in terms of column vectors

Invariance of rank under elementary row operations

Short Courses. For the further discussion in the next sections, it suffices to define linear independence and rank.

Comments on Rank and Vector Spaces

Of the three possible equivalent definitions of **rank,**

(i) By row vectors (our definition),

(ii) By column vectors (our Theorem 1),

(iii) By submatrices with nonzero determinant (Sec. 6.6),

the first seems to be most practical in our context.

Introducing **vector spaces** here, rather than in Sec. 6.1, we have the advantage that the student immediately sees an application (row and column spaces). Vector spaces in full generality follow in Sec. 6.8.

SOLUTIONS TO PROBLEM SET 6.4, page 336

2. Linearly dependent

4. Linearly dependent (four vectors in R^3!)

6. Linearly dependent (one is the zero vector!)

8. Linearly dependent

10. 2 **12.** 3 **14.** 3 **16.** 3

18. Yes when $k = 0$, dimension 2, basis $[1 \quad 0 \quad 0]$, $[0 \quad 1 \quad -4]$. No for any other value of k

20. No, because of the inequality

22. Yes, dimension 2, basis $\mathbf{e}_{(n-1)}$ and $\mathbf{e}_{(n)}$ (the last two vectors of the standard basis)

24. Yes, dimension 1, basis $[5 \quad -4 \quad -23]$, as follows by first considering the second equation and then the first

26. TEAM PROJECT. (b) $\mathbf{B}^T\mathbf{A}^T = (\mathbf{AB})^T$ and rank is invariant under transposition. The other two statements follow from the definition of rank and Theorem 1.

Parts (c) and (d) are proved in Ref. [B2] listed in Appendix 1. Equality in (d) occurs for $\mathbf{A} = \mathbf{B} = \mathbf{I}$, for instance.

28. $[2 \quad -1]$, $[4 \quad -1 \quad 3]^T$

SECTION 6.5. Solutions of Linear Systems: Existence, Uniqueness, General Form, page 338

Purpose. The student should see that the totality of solutions (including the existence and uniqueness) can be characterized in terms of the ranks of the coefficient matrix and the augmented matrix.

Main Content, Important Concepts

Augmented matrix

Necessary and sufficient conditions for the existence of solutions

Implications for homogeneous systems

rank $\mathbf{A}$ + nullity $\mathbf{A} = n$

Background Material. Rank (Sec. 6.4)

Short Courses. Brief discussion of the first two theorems, illustrated by some simple examples.

Comments on Content

This section should make the student aware of the great importance of rank. It may be good to have students memorize the condition

$$\text{rank } \mathbf{A} = \text{rank } \tilde{\mathbf{A}}$$

for the existence of solutions.

Students familiar with differential equations may be reminded of the analog of Theorem 4 (see Sec. 2.8).

This section may also provide a good opportunity to point to the roles of existence and uniqueness problems throughout mathematics (and to the distinction between the two).

SECTION 6.6. Determinants. Cramer's Rule, page 341

Purpose. The first part of this section (on second- and third-order determinants) is mainly for reference in other chapters. The main body of the section concerns those properties of nth-order determinants that are needed in practical work, and Cramer's rule.

so that we obtain

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

The sphere through the given points is $x^2 + y^2 + (z - 1)^2 = 16$.

(e) For a general conic section the equation is

$$ax^2 + bxy + cy^2 + dx + ey + f \cdot 1 = 0,$$

so that we get

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0.$$

SECTION 6.7. Inverse of a Matrix. Gauss–Jordan Elimination, page 350

Purpose. To familiarize the student with the concept of the inverse $\mathbf{A}^{-1}$ of a square matrix $\mathbf{A}$, its conditions for existence, and its computation.

Main Content, Important Concepts

$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

Nonsingular and singular matrices

Existence of $\mathbf{A}^{-1}$ and rank

Gauss–Jordan elimination

$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$

Cancellation law

$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}$

Short Courses. Theorem 1 without proof, Gauss–Jordan elimination, formulas (4*) and (7).

Comments on Content

Although in this chapter we are not concerned with operations count (Chap. 18), it would make no sense to first blindfold the student by using Gauss–Jordan for solving $\mathbf{Ax} = \mathbf{b}$ and then later in numerical analysis correct the false impression by explaining why Gauss elimination is better because back substitution needs fewer operations than the diagonalization of a triangular matrix. Thus Gauss–Jordan should be applied only when $\mathbf{A}^{-1}$ is needed.

The "unusual" properties of matrix multiplication, briefly mentioned in Sec. 6.2 can now be explored systematically by the use of rank and inverse.

Formula (4*) is worth memorizing.

Main Content, Important Concepts

Second- and third-order determinants

nth-order determinants

General properties of determinants

Rank in terms of determinants (Theorem 3)

Cramer's rule for solving linear systems by determinants (Theorem 4)

General Comments on Determinants

Our definition of a determinant seems more practical than that in terms of permutations (because it immediately gives those general properties), at the expense of the proof that our definition is unambiguous (see the proof in Appendix 4).

General properties are given for order n, from which they can be easily seen for $n = 3$ when needed.

The importance of determinants has decreased with time, but will remain basic in eigenvalue problems (characteristic determinants), differential equations (Wronskians!), integration and transformations (Jacobians!), and other areas of practical interest.

SOLUTIONS TO PROBLEM SET 6.6, page 349

6. 1 **8.** $-42{,}640$ **10.** 0

12. -64 **14.** 1 **16.** 2

18. $x = 2, y = -3, z = 8$

20. TEAM PROJECT. (b) For a plane the equation is $ax + by + cz + d \cdot 1 = 0$, so that we get the determinantal equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

The plane is $3x + 4y - 2z = 5$.

(c) For a circle the equation is

$$a(x^2 + y^2) + bx + cy + d \cdot 1 = 0,$$

so that we get

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

The circle is $x^2 + y^2 - 4x - 2y = 20$.

(d) For a sphere the equation is

$$a(x^2 + y^2 + z^2) + bx + cy + dz + e \cdot 1 = 0,$$

SOLUTIONS TO PROBLEM SET 6.7, page 357

2. $\begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

4. Note that due to the special form of the given matrix, the 2×2 minor in the right lower corner of the inverse has the form of the inverse of a 2×2 matrix; the inverse is

$$\begin{bmatrix} -1/7 & 0 & 0 \\ 0 & 5 & -13 \\ 0 & -3 & 8 \end{bmatrix}.$$

6. The entries of the inverse are the same as for a diagonal matrix, but their position on the other diagonal is different. The inverse is

$$\begin{bmatrix} 0 & 0 & 5/2 \\ 0 & -5 & 0 \\ 10/3 & 0 & 0 \end{bmatrix}.$$

8. The given matrix is singular. It is interesting that this is not the case for the 2×2 matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

10. Multiply $\mathbf{I} = (\mathbf{A}^2)^{-1}\mathbf{A}^2$ by $\mathbf{A}^{-1}$ from the right, $\mathbf{A}^{-1} = (\mathbf{A}^2)^{-1}\mathbf{A}$, and this result again by $\mathbf{A}^{-1}$ from the right.

12. We obtain

$$\begin{aligned} \mathbf{I} = \mathbf{I}^{\mathsf{T}} &= (\mathbf{A}\mathbf{A}^{-1})^{\mathsf{T}} = (\mathbf{A}^{-1}\mathbf{A})^{\mathsf{T}} \\ &= (\mathbf{A}^{-1})^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}}(\mathbf{A}^{-1})^{\mathsf{T}}. \end{aligned}$$

This shows that the inverse of $\mathbf{A}^{\mathsf{T}}$ must be $(\mathbf{A}^{-1})^{\mathsf{T}}$, as we wanted to prove.

14. Use (1), with $\mathbf{A}$ replaced by $\mathbf{C}$, and set $\mathbf{C} = \mathbf{A}^{-1}$.

16. $\begin{bmatrix} -\frac{1}{20} & 0 & \frac{1}{5} \\ 0 & \frac{1}{2} & 0 \\ \frac{3}{10} & 0 & -\frac{1}{5} \end{bmatrix}$ **18.** $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 3 & -1 & 0 \\ -\frac{23}{8} & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$ **20.** $\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{12} \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{24} \end{bmatrix}$

SECTION 6.8. Vector Spaces. Inner Product Spaces. Linear Transformations. *Optional,* page 358

Purpose. In this optional section we extend our earlier discussion of vector spaces R^n and C^n, define inner product spaces, and explain the role of matrices in linear transformations of R^n into R^m.

Main Content, Important Concepts

Real vector space, complex vector space

Linear independence, dimension, basis

Inner product space

Linear transformation of R^n into R^m

Background Material. Vector spaces R^n and C^n (Sec. 6.4), inner product (Sec. 6.2).

Comments on Content

The student is supposed to see and comprehend how concrete models (R^n and C^n; the inner product for vectors) lead to abstract concepts, defined by axioms resulting from basic properties of those models. Because of the level and general objective of this chapter, we have to restrict our discussion to the illustration and explanation of the abstract concepts in terms of some simple typical examples.

Most essential from the viewpoint of matrices is our discussion of *linear transformations,* which in a more theoretically oriented course of a higher level would occupy a more prominent position.

Comment on Footnote 12

Hilbert's work was fundamental to various areas in mathematics; roughly speaking, he worked on number theory 1893–1898, foundations of geometry 1898–1902, integral equations 1902–1912, physics 1910–1922, and logic and foundations of mathematics 1922–1930. Closest to our interests here is the development in integral equations, as follows. In 1870 Carl Neumann (Sec. 4.6) had the idea of solving the Dirichlet problem for the Laplace equation (Sec. 9.8) by converting it to an integral equation. This created general interest in integral equations. In 1896 Vito Volterra (1860–1940) developed a general theory of these equations, followed by Ivar Fredholm (1866–1927) in 1900–1903, whose papers caused great excitement, and Hilbert since 1902. This gave the impetus to the development of inner product and Hilbert spaces and operators defined on them. These spaces and operators and their spectral theory have found basic applications in quantum mechanics since 1927. Hilbert's great interest in mathematical physics is documented by Ref. [4], a classic full of ideas that are of interest to the mathematical work of the engineer. For more details, see G. Birkhoff and E. Kreyszig. The establishment of functional analysis. *Historia Mathematica* **11** (1984), pp. 258–321.

SOLUTIONS TO PROBLEM SET 6.8, page 364

2. No; nonnegativity is not preserved under scalar multiplication.

4. Dimension 2. Basis $[\cos x \quad \sin x]$. Further examples from differential equations can easily be presented to students familiar with these equations. We did not mention this explicitly, to keep chapters independent.

6. Dimension 6, basis

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\quad \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},$$

$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

8. Dimension 4, basis

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10. No

12. If another such representation with coefficients k_j would also hold, subtraction would give $\Sigma(c_j - k_j)\mathbf{a}_j = \mathbf{0}$, hence $c_j - k_j = 0$, because of the linear independence. This shows the uniqueness.

14. $\sqrt{77}$ **16.** $\sqrt{38}$ **18.** $\sqrt{62}$ **20.** $\pm[0.8 \quad 0.6]^{\mathsf{T}}$

22. $0.8 + 3.9 + 11.0 = 15.7 < 2.58650\sqrt{38} = 15.944$

24. $\| [3 \quad 4]^{\mathsf{T}} \|^2 + \| [1 \quad -2]^{\mathsf{T}} \|^2 = 25 + 5 = 2(5 + 10)$

26. $x_1 = 5y_1 - y_2$
$x_2 = 3y_1 - y_2$

28. $x_1 = 0.25y_1 - 0.1y_3$
$x_2 = y_2 - 0.8y_3$
$x_3 = 0.2y_3$

30. $x_1 = y_1$
$x_2 = y_2 \cos\theta - y_3 \sin\theta$
$x_3 = y_2 \sin\theta + y_3 \cos\theta$

SOLUTIONS TO CHAPTER 6 REVIEW, page 365

12. $\begin{bmatrix} 1 & 18 & 13 \\ -6 & -8 & 2 \\ -1 & 7 & 7 \end{bmatrix}$ **14.** $\begin{bmatrix} 19 & 1 & -22 \\ 1 & 21 & 15 \\ -22 & 15 & 38 \end{bmatrix}$ **16.** $\begin{bmatrix} -2 & -12 & -12 \\ -12 & 16 & -9 \\ -12 & -9 & -14 \end{bmatrix}$

18. $\begin{bmatrix} 0 & 4 & 1 \\ -4 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}$ **20.** $\begin{bmatrix} -9 \\ -34 \\ -13 \end{bmatrix}$ **22.** -5

24. $[9 \quad 34 \quad 13]$ **26.** $x = 2, y = -1, z = 4$ **28.** No solution

30. $x = 0, y = 2, z = -3$ **32.** $x = 1, z = 3y + 2$ **34.** $x = 2 + \frac{13}{2}y - \frac{3}{2}z$

36. 2, 3 **38.** 3, 3 **40.** 2, 2

42. A 3×3 skew-symmetric matrix is always singular. Hence any product involving $\mathbf{B}$ or $\mathbf{B}^{\mathsf{T}}$ is singular, by the theorem on the determinants of products of matrices (Sec. 6.7, Theorem 4).

44. $\begin{bmatrix} 2.5 & -1 & -4.5 \\ 3 & 1 & -1 \\ -3.5 & -3 & 8.5 \end{bmatrix}$

46. $I_1 = 12, I_2 = 4, I_3 = 16$ [amps]

48. $I_1 = 4, I_2 = 5, I_3 = 1$ [amps]

50. By Kirchhoff's current law, $i_1 = i_2 + u_2/Z_2$. From this, Kirchhoff's voltage law, and Ohm's law,

$$u_1 = Z_1 i_1 + u_2 = Z_1\left(i_2 + \frac{1}{Z_2}u_2\right) + u_2.$$

This gives the indicated matrix.

CHAPTER 7 Linear Algebra: Matrix Eigenvalue Problems

This chapter is new. Prerequisite is some familiarity with the notion of a matrix and with the two algebraic operations for matrices. Otherwise the chapter is independent of Chap. 6, so that it can be used for teaching eigenvalue problems and their applications, without first going through the material in Chap. 6.

SECTION 7.1. Eigenvalues, Eigenvectors, page 371

Purpose. To familiarize the student with the determination of eigenvalues and eigenvectors of real matrices and to give a first impression of what one can expect (multiple eigenvalues, complex eigenvalues, etc.).

Main Content, Important Concepts

Eigenvalue, eigenvector

Determination of eigenvalues from the characteristic equation

Determination of eigenvectors

Algebraic and geometric multiplicity, defect

Comments on Content

To maintain undivided attention on the basic concepts and techniques, all the examples in this section are formal, and typical applications are put into a separate section (Sec. 7.2).

The distinction between the algebraic and geometric multiplicity is mentioned in this early section, and the idea of a *basis of eigenvectors* could perhaps be mentioned briefly in class, whereas a thorough discussion of this in a later section (Sec. 7.5) will profit from the increased experience with eigenvalue problems, which the student will have gained at that later time.

The possibility of *normalizing* any eigenvector is mentioned in Theorem 2, but this will be of greater interest to us only in connection with orthonormal or unitary systems (Sec. 7.4).

In our present work we find eigen*values* first and are then left with the much simpler task of determining corresponding eigen*vectors*. Numerical work (Secs. 18.6–18.9) may proceed in the opposite order, but to mention this here would perhaps just confuse the student.

SOLUTIONS TO PROBLEM SET 7.1, page 375

2. 0, any nonzero vector

4. $-a, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, a, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

6. $-3i, \begin{bmatrix} 1 \\ -i \end{bmatrix}, 3i, \begin{bmatrix} 1 \\ i \end{bmatrix}$. This result is typical because the matrix is skew-symmetric. We discuss this in Sec. 7.3.

8. $0.8 + 0.6i, \begin{bmatrix} 1 \\ -i \end{bmatrix}, 0.8 - 0.6i, \begin{bmatrix} 1 \\ i \end{bmatrix}$. This is typical, as we shall see in Sec. 7.3; namely, the matrix is orthogonal, its eigenvalues have absolute value 1, and its determinant has value 1.

10. $-15, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, -15, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, 25, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

12. $a, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, a - \sqrt{2}, \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}, a + \sqrt{2}; \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

14. $-2, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. $\lambda = 0$ has algebraic multiplicity 2 and geometric multiplicity 1 (just as $\lambda = 3$ in Prob. 13), so that we have no basis of eigenvectors.

16. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues are i and $-i$. Corresponding eigenvectors are complex, indicating that there is no direction that is preserved under a rotation.

18. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. An eigenvalue is 1, with eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, indicating that every point in the xy-plane is mapped onto itself. The other eigenvalue is -1, with eigenvector $[0 \quad 0 \quad 1]^T$, indicating that every point z_1 on the z-axis is mapped onto its negative $-z_1$.

20. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The eigenvalue 1 with eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ indicates that every point in the plane $y = x$ is mapped onto itself. The other eigenvalue 0 with eigenvector $[1 \quad -1 \quad 0]^T$ indicates that any point on the line $y = -x$, $z = 0$ (which is perpendicular to the plane $y = x$) is mapped onto the origin. The student should perhaps make a sketch to see what is going on geometrically.

SECTION 7.2. Some Applications of Eigenvalue Problems, page 376

Purpose. Matrix eigenvalue problems are of greatest importance in physics, engineering, geometry, etc., and the applications in this section and in the problem set are supposed to give the student at least some impression of this fact.

Main Content

Applications of eigenvalue problems in

Elasticity theory (Example 1),

Probability theory (Example 2),

Biology (Example 3),

Mechanical vibrations (Example 4).

Short Courses. Of course, this section can be omitted, for reasons of time, or one or two of the examples can be considered quite briefly.

Comments on Content

The examples in this section have been selected from the viewpoint of modest prerequisites, so that not too much time will be needed to set the scene.

Example 4 illustrates why real matrices can have complex eigenvalues (as mentioned before, in Sec. 7.1), and why these eigenvalues are physically meaningful. (For students familiar with systems of differential equations, one can easily pick further examples from Chap. 3.)

SOLUTIONS TO PROBLEM SET 7.2, page 379

2. Eigenvalues and eigenvectors are 1.6, $[1 \quad -1]^T$ and 2.4, $[1 \quad 1]^T$. These vectors are orthogonal, as is typical of a symmetric matrix. Directions are $-45°$ and $45°$, respectively.

4. 0.5, $[1 \quad -1]^T$; 1.5, $[1 \quad 1]^T$. Orthogonality as in Prob. 2. Directions $-45°$ and $45°$, respectively.

6. 2, $[1 \quad 1]^T$; $\frac{1}{2}$, $[1 \quad -1]^T$. Directions $45°$ and $-45°$, respectively.

8. $[1 \quad 1 \quad 1]^T$. This could be seen without calculation because the matrix also has column sums equal to 1, which is not the case in general.

10. The growth rate is 2. The other two eigenvalues are not needed; they could be determined by dividing the characteristic polynomial by $\lambda - 2$; they are $-1 \pm \sqrt{0.6}$.

12. Growth rate 3. The other eigenvalues are -0.247004 and -2.753. These are not needed.

14. $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}^T$, and $\mathbf{A}^T$ has row sums 1, so that it has the eigenvalue 1 with eigenvector $\mathbf{x} = [1 \cdots 1]^T$.

Leontief is a leader in the development and application of quantitative methods in empirical economical research, using genuine data from the economy of the United States to provide, in addition to the "closed model" of Prob. 13 (where the producers consume the whole production), "open models" of various situations of production and consumption, including import, export, taxes, capital gains and losses, etc. See W. W. Leontief, *The Structure of the American Economy 1919–1939* (Oxford: Oxford University Press, 1951), H. B. Cheney and P. G. Clark, *Interindustry Economics* (New York: Wiley, 1959).

16. TEAM PROJECT. (a) Because a polynomial with real coefficients (in our case, the characteristic polynomial) has real or complex conjugate zeros.

(b) $\mathbf{A}^{-1}$ exists if and only if $\det \mathbf{A} \neq 0$, but $\det \mathbf{A} = \lambda_1 \lambda_2 \cdots \lambda_n$, as follows from the product representation

$$D(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n),$$

namely,

$$\det \mathbf{A} = (-1)^n (-\lambda_1)(-\lambda_2) \cdots (-\lambda_n) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

(c) This follows by comparing the coefficient of λ^{n-1} in the expansion of $D(\lambda)$ with that obtained from the product representation.

(d) $\mathbf{A}\mathbf{x}_j = \lambda_j \mathbf{x}_j \; (\mathbf{x}_j \neq \mathbf{0}), \quad (\mathbf{A} - k\mathbf{I})\mathbf{x}_j = \lambda_j \mathbf{x}_j - k\mathbf{x}_j = (\lambda_j - k)\mathbf{x}_j$

(e) The first statement follows from

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}, \quad (k\mathbf{A})\mathbf{x} = k(\mathbf{A}\mathbf{x}) = k(\lambda \mathbf{x}) = (k\lambda)\mathbf{x},$$

the second by induction and multiplication of $\mathbf{A}^k \mathbf{x}_j = \lambda_j^k \mathbf{x}_j$ by $\mathbf{A}$ from the left.

(f) From $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$ ($\mathbf{x}_j \neq \mathbf{0}$) and (e) follows $k_p\mathbf{A}^p\mathbf{x}_j = k_p\lambda_j{}^p\mathbf{x}_j$ and $k_q\mathbf{A}^q\mathbf{x}_j = k_q\lambda_j{}^q\mathbf{x}_j$ ($p \geqq 0$, $q \geqq 0$, integer). Adding on both sides, we see that $k_p\mathbf{A}^p + k_q\mathbf{A}^q$ has the eigenvalue $k_p\lambda_j{}^p + k_q\lambda_j{}^q$. From this the statement follows.

(g) $\det(\mathbf{L} - \lambda\mathbf{I}) = -\lambda^3 + l_{12}l_{21}\lambda + l_{13}l_{21}l_{32} = 0$. Hence $\lambda \neq 0$. If all three eigenvalues are real, at least one is positive since trace $\mathbf{L} = 0$. The only other possibility is $\lambda_1 = a + ib$, $\lambda_2 = a - ib$, λ_3 real (except for the numbering of the eigenvalues). Then $\lambda_3 > 0$ because [see (b)]

$$\lambda_1\lambda_2\lambda_3 = (a^2 + b^2)\lambda_3 = \det \mathbf{L} = l_{13}l_{21}l_{32} > 0.$$

SECTION 7.3. Symmetric, Skew-Symmetric, and Orthogonal Matrices, page 381

Purpose. To introduce the student to the three most important classes of real square matrices and their general properties and eigenvalue theory.

Main Content, Important Concepts

The eigenvalues of a symmetric matrix are real.

The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.

The eigenvalues of an orthogonal matrix have absolute value 1.

Further properties of orthogonal matrices

Comments on Content

The student should memorize the preceding three statements on the locations of eigenvalues as well as the basic properties of orthogonal matrices (orthonormality of row vectors and of column vectors, invariance of inner product, determinant equal to 1 or -1).

Furthermore, it may be good to emphasize that, since the eigenvalues of an orthogonal matrix may be complex, so may be the eigenvectors. Similarly for skew-symmetric matrices. Both cases are simultaneously illustrated by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{with eigenvectors} \quad \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

corresponding to the eigenvalues i and $-i$, respectively.

SOLUTIONS TO PROBLEM SET 7.3, page 384

2. Skew-symmetric if $a = 0$, symmetric if $b = 0$, orthogonal if $a^2 + b^2 = 1$. Eigenvalues $a \pm ib$

4. Orthogonal (a rotation about the x-axis through an angle θ). Eigenvalues 1 and $\cos\theta \pm i \sin\theta$

6. Symmetric (for real a and k). Eigenvalues $a - k$ (of algebraic and geometric multiplicities 2 when $k \neq 0$) and $a + 2k$

8. Let $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ ($\mathbf{x} \neq \mathbf{0}$), $\mathbf{A}\mathbf{y} = \mu\mathbf{y}$ ($\mathbf{y} \neq \mathbf{0}$). Then $(\mathbf{A}\mathbf{x})^\mathsf{T} = \mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T} = \mathbf{x}^\mathsf{T}\mathbf{A} = \lambda\mathbf{x}^\mathsf{T}$. Thus $\lambda\mathbf{x}^\mathsf{T}\mathbf{y} = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{y} = \mathbf{x}^\mathsf{T}\mu\mathbf{y} = \mu\mathbf{x}^\mathsf{T}\mathbf{y}$. Hence, if $\lambda \neq \mu$, then $\mathbf{x}^\mathsf{T}\mathbf{y} = 0$, which proves orthogonality.

10. Yes, for instance

$$\begin{bmatrix} a & \sqrt{1-a^2} & 0 \\ \sqrt{1-a^2} & -a & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $-1 \leqq a \leqq 1$.

12. No for 3×3, yes for 4×4, no for 5×5. For 3×3,

$$\det \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = -abc + bac = 0.$$

$\det \mathbf{A} = \det(\mathbf{A}^\mathsf{T}) = \det(-\mathbf{A}) = (-1)^n \det \mathbf{A} = 0$ if $n = 3, 5, \cdots$.

14. **(a)** $\mathbf{A}^\mathsf{T} = \mathbf{A}^{-1}$, $\mathbf{B}^\mathsf{T} = \mathbf{B}^{-1}$, $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T} = \mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$. Also $(\mathbf{A}^{-1})^\mathsf{T} = (\mathbf{A}^\mathsf{T})^{-1} = (\mathbf{A}^{-1})^{-1}$. In terms of rotations it means that the composite of rotations and the inverse of a rotation are rotations.

(b) The inverse is

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

(c) To a rotation of 16.26°. No limit. For a student unfamiliar with complex numbers this may require some thought.

(d) Limit 0, approach along some spiral.

(e) The matrix is obtained by using familiar values of cosine and sine,

$$\mathbf{A} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}.$$

SECTION 7.4. Complex Matrices: Hermitian, Skew-Hermitian, Unitary, page 385

Purpose. This section is devoted to the three most important classes of complex matrices and corresponding forms and eigenvalue theory.

Main Content, Important Concepts

Hermitian and skew-Hermitian matrices

Unitary matrices, unitary systems

Location of eigenvalues (Fig. 146)

Quadratic forms, their symmetric coefficient matrix

Hermitian and skew-Hermitian forms

Background Material. Section 7.3, which the present section generalizes. The prerequisites on complex numbers are very modest, so that students will hardly need any extra help in that respect.

Short Courses. This section can be omitted.

Comments on Content

This is the first time in this chapter that the student meets with complex matrices. The material is arranged so that the analogy of properties and proofs to those in Sec. 7.3 will be apparent.

The importance of these matrices results from quantum mechanics as well as from mathematics itself (e.g., from unitary transformations, product representations of nonsingular matrices $\mathbf{A} = \mathbf{UH}$, $\mathbf{U}$ unitary, $\mathbf{H}$ Hermitian, etc.).

The determinant of a unitary matrix (see Theorem 4) may be complex. For example, the matrix

$$\mathbf{A} = \frac{1+i}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is unitary and has

$$\det \mathbf{A} = i.$$

SOLUTIONS TO PROBLEM SET 7.4, page 390

2. $[1-3i \quad 5]^{\mathsf{T}}$, $[1-3i \quad -2]^{\mathsf{T}}$

4. $[1 \quad 1]^{\mathsf{T}}$, $[1 \quad -1]^{\mathsf{T}}$

6. Skew-Hermitian; $-i$, $[-1+i \quad 2]^{\mathsf{T}}$; $2i$, $[1-i \quad 1]^{\mathsf{T}}$

8. Skew-Hermitian; i, $[0 \quad 1 \quad 0]^{\mathsf{T}}$; $3i$, $[-1 \quad 0 \quad 1]^{\mathsf{T}}$; $5i$, $[1 \quad 0 \quad 1]^{\mathsf{T}}$

10. Hermitian; -2, $[i \quad -1-i \quad 1]^{\mathsf{T}}$; 0, $[-i \quad 0 \quad 1]^{\mathsf{T}}$; 2, $[i \quad 1+i \quad 1]^{\mathsf{T}}$

12. PROJECT. **(a)** For $\mathbf{A}$, $\mathbf{B}$ unitary, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \overline{\mathbf{B}}^{\mathsf{T}}\overline{\mathbf{A}}^{\mathsf{T}} = (\overline{\mathbf{AB}})^{\mathsf{T}}$.

We prove the statement about the inverse. Let $\mathbf{A}$ be unitary. Set $\mathbf{A}^{-1} = \mathbf{B}$. Then $\mathbf{B}^{\mathsf{T}} = (\mathbf{A}^{-1})^{\mathsf{T}} = (\mathbf{A}^{\mathsf{T}})^{-1} = (\overline{\mathbf{A}}^{-1})^{-1} = \overline{\mathbf{B}}^{-1}$. Thus $\mathbf{B}^{-1} = \overline{\mathbf{B}}^{\mathsf{T}}$.

(c) $\mathbf{A}\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^2$ if $\mathbf{A}$ is Hermitian, $-\mathbf{A}^2$ if $\mathbf{A}$ is skew-Hermitian, $\mathbf{AA}^{-1} = \mathbf{I}$ if $\mathbf{A}$ is unitary. Commutability is now obvious.

(d) $\mathbf{A} = \mathbf{H} + \mathbf{S}$, $\overline{\mathbf{A}}^{\mathsf{T}} = \overline{\mathbf{H}}^{\mathsf{T}} + \overline{\mathbf{S}}^{\mathsf{T}} = \mathbf{H} - \mathbf{S}$, hence

$$\mathbf{A}\overline{\mathbf{A}}^{\mathsf{T}} = (\mathbf{H}+\mathbf{S})(\mathbf{H}-\mathbf{S}) = \mathbf{H}^2 - \mathbf{HS} + \mathbf{SH} - \mathbf{S}^2.$$

Also

$$\overline{\mathbf{A}}^{\mathsf{T}}\mathbf{A} = (\mathbf{H}-\mathbf{S})(\mathbf{H}+\mathbf{S}) = \mathbf{H}^2 + \mathbf{HS} - \mathbf{SH} - \mathbf{S}^2.$$

These two expressions are equal if and only if

$$-\mathbf{HS} + \mathbf{SH} = \mathbf{HS} - \mathbf{SH}.$$

This implies that $\mathbf{HS} = \mathbf{SH}$, as claimed.

(e) For instance,

$$\begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}$$

is not normal. A normal matrix that is not Hermitian, skew-Hermitian, or unitary is obtained if we take a unitary matrix and multiply it by 2 or some other real factor different from ± 1.

14. $\begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & -9 \end{bmatrix}$ **16.** $\begin{bmatrix} 1 & -2 & 3 & -1 \\ -2 & 4 & -6 & 2 \\ 3 & -6 & 9 & -3 \\ -1 & 2 & -3 & 1 \end{bmatrix}$ **18.** $\begin{bmatrix} 1 & -1 & 4 & 0 \\ -1 & -3 & -4 & 4 \\ 4 & -4 & 16 & 0 \\ 0 & 4 & 0 & -4 \end{bmatrix}$

20. Skew-Hermitian, $6i$

22. Hermitian, $a|x_1|^2 + 2\,\mathrm{Re}\,[(b+ic)\bar{x}_1 x_2] + k|x_2|^2$

24. Hermitian, 4

SECTION 7.5. Similarity of Matrices. Basis of Eigenvectors. Diagonalization, page 392

Purpose. This section exhibits the role of bases of eigenvectors in connection with linear transformations and contains theorems of great practical importance in connection with eigenvalue problems, notably Theorems 1, 4, 5.

Main Content, Important Concepts

Similar matrices have the same spectrum (Theorem 1).

Bases of eigenvectors (Theorems 3, 4)

Diagonalization of matrices (Theorem 5)

Principal axes transformation of forms

Short Courses. Complete omission of this section or restriction to a short look at Theorems 1 and 5.

Comments on Content

Theorem 1 on similar matrices has various applications in the design of numerical methods (Chap. 18), which often use subsequent similarity transformations to tridiagonalize or (nearly) diagonalize matrices on the way to approximations of eigenvalues and eigenvectors. The matrix **X** of eigenvectors [see (5)] also occurs quite frequently in that context.

Theorem 4 is another result of fundamental importance in many applications, for instance, in those methods for numerically determining eigenvalues and eigenvectors. Its proof is substantially more difficult than the other proofs given in this chapter.

SOLUTIONS TO PROBLEM SET 7.5, page 397

2. $\hat{\mathbf{A}} = \begin{bmatrix} 3.008 & -0.544 \\ 5.456 & 6.992 \end{bmatrix}$; $\lambda = 4$, $\mathbf{y} = \begin{bmatrix} -17 \\ 31 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 25 \\ 25 \end{bmatrix}$;

$\lambda = 6$, $\mathbf{y} = \begin{bmatrix} -2 \\ 11 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

4. $\hat{\mathbf{A}} = \begin{bmatrix} -29 & 20 \\ -42 & 29 \end{bmatrix}$; $\lambda = -1$, $\mathbf{y} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$;

$\lambda = 1$, $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

6. $\hat{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 3 \\ 0 & 2 & 2 \end{bmatrix}$; $\lambda = 1$, $\mathbf{y} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$;

$\lambda = 8$, $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

8. $[(1+i)/2 \quad 1/\sqrt{2}]^T$, $[(1+i)/2 \quad -1/\sqrt{2}]^T$

10. $\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}2 & 0\\0 & -5\end{bmatrix}$

12. $\begin{bmatrix}7\\13\end{bmatrix}, \begin{bmatrix}11\\-1\end{bmatrix}, \begin{bmatrix}100 & 0\\0 & -50\end{bmatrix}$

14. $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}\frac{1}{2} & 0 & 0\\0 & 2 & 0\\0 & 0 & -1\end{bmatrix}$

16. Hyperbola $52y_1^2 - 39y_2^2 = 156$, hence $4y_1^2 - 3y_2^2 = 12$; $x_1 = (2y_1 + 3y_2)/\sqrt{13}$, $x_2 = (3y_1 - 2y_2)/\sqrt{13}$

18. Orthogonal straight lines $25y_1^2 + 50y_2^2 = 0$, that is, the coordinate axes of the y-system, $x_1 = 0.6y_1 - 0.8y_2$, $x_2 = 0.8y_1 + 0.6y_2$

20. Ellipse $y_1^2 + 16y_2^2 = 16$, $x_1 = (2y_1 + y_2)/\sqrt{5}$, $x_2 = (-y_1 + 2y_2)/\sqrt{5}$

22. PROJECT. **(a)** This follows immediately from the product representation of the characteristic polynomial of **A**.

(b) $\mathbf{C} = \mathbf{AB}$, $c_{11} = \sum_{l=1}^{n} a_{1l}b_{l1}$, $c_{22} = \sum_{l=1}^{n} a_{2l}b_{l2}$, etc. Now take the sum of these n sums. Furthermore, trace **BA** is the sum of

$$\tilde{c}_{11} = \sum_{m=1}^{n} b_{1m}a_{m1}, \cdots, \tilde{c}_{nn} = \sum_{m=1}^{n} b_{nm}a_{mn},$$

involving the same n^2 terms as those in the double sum of trace **AB**.

(c) By multiplications from the right and from the left we readily obtain

$$\tilde{\mathbf{A}} = \mathbf{P}^2\hat{\mathbf{A}}\mathbf{P}^{-2}$$

(d) Interchange the corresponding eigenvectors (columns) in the matrix **X** in (5).

SOLUTIONS TO CHAPTER 7 REVIEW, page 398

12. $-4, \begin{bmatrix}2\\3\end{bmatrix}; 4, \begin{bmatrix}1\\2\end{bmatrix}$

14. $1, \begin{bmatrix}1\\0\\0\end{bmatrix}; 0, \begin{bmatrix}1\\1\\1\end{bmatrix}; 3, \begin{bmatrix}0\\0\\1\end{bmatrix}$

16. $-3, \begin{bmatrix}1\\1\\-1\end{bmatrix}; 3, \begin{bmatrix}1\\1\\2\end{bmatrix}; 15, \begin{bmatrix}2\\-1\\1\end{bmatrix}$

18. $\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}18 & 0 & 0\\0 & 9 & 0\\0 & 0 & -9\end{bmatrix}$

20. Hyperbola $-10y_1^2 + 5y_2^2 = 10$, $x_1 = (y_1 + 2y_2)/\sqrt{5}$, $x_2 = (-2y_1 + y_2)/\sqrt{5}$

22. Straight lines $y_2 = \pm y_1$, resulting from $13(-y_1^2 + y_2^2) = 13(y_2 + y_1)(y_2 - y_1) = 0$; $x_1 = (2y_1 + 3y_2)/\sqrt{13}$, $x_2 = (-3y_1 + 2y_2)/\sqrt{13}$

24. Eigenvalues and eigenvectors are

$$S_x: \quad -1, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad 1, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S_y: \quad -1, \quad \begin{bmatrix} 1 \\ -i \end{bmatrix}; \quad 1, \quad \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$S_z: \quad -1, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad 1, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

CHAPTER 8 Vector Differential Calculus. Grad, Div, Curl

Changes

These are minor. In Sec. 8.6, tangential and normal accelerations are discussed in a more concrete fashion. The section on grad, div, and curl in curvilinear coordinates has been omitted; this material can now be found in Appendix A3.4, since it is mainly for reference.

SECTION 8.1. Vector Algebra in 2-Space and 3-Space, page 401

Purpose. We introduce vectors in 3-space given geometrically by (families of parallel) directed segments or algebraically by ordered triples of real numbers, and we define addition of vectors and scalar multiplication (multiplication of vectors by numbers).

Main Content, Important Concepts

Vector, norm (length), unit vector, components

Addition of vectors, scalar multiplication

Vector space R^3, linear independence, basis

Comments on Content

Our discussions in the whole chapter will be independent of Chap. 6, and there will be no more need for writing vectors as columns and for distinguishing between row and column vectors. Our notation $\mathbf{a} = [a_1, a_2, a_3]$ is compatible with that in Chap. 6. Engineers seem to like both notations

$$\mathbf{a} = [a_1, a_2, a_3] = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

preferring the first for "short" components and the second in the case of longer expressions.

The student is supposed to understand that the whole vector algebra (and vector calculus) has resulted from applications, with concepts that are practical, that is, they are "made to measure" for standard needs and situations; thus, in this section, the two algebraic operations resulted from forces (forming resultants and changing magnitudes of forces); similarly in the next sections. The restrictions to three dimensions (as opposed to n dimensions in the previous two chapters) allows us to "visualize" concepts, relations, and results and to give geometrical explanations and interpretations.

On a higher level, the equivalence of the geometric and the algebraic approach (Theorem 1) would require a consideration of how the various triples of numbers for the various choices of coordinate systems must be related (in terms of coordinate transformations) for a vector to have a norm and direction independent of the choice of coordinate systems.

SOLUTIONS TO PROBLEM SET 8.1, page 407

2. $-1, 4, 3; \sqrt{26}$ **4.** $-10, 2, -6; \sqrt{140}$ **6.** $a, b, c; \sqrt{a^2 + b^2 + c^2}$

8. $-2, 0, -2; \sqrt{8}$ **10.** $(0, 0, 1); \sqrt{13}$ **12.** $(3, -1, -2); 0$

14. $(3, -1, 6); \sqrt{46}$ **16.** $[-3, 2, -1], [9, -6, 3], [-\frac{3}{2}, 1, -\frac{1}{2}]$
18. $\sqrt{11}, \sqrt{14} + 3$ **20.** $(1/\sqrt{14})\ [3, -2, 1], [0, 1, 0]$
22. $[27, -19, 2]$ **24.** $[8, -3, 8]; \sqrt{137}$ **26.** $[0, -10, 0]$
28. $[12, 0, 16]; 20$
30. $2 \leqq |\mathbf{p} + \mathbf{q}| \leqq 10,\ 12 \leqq |4\mathbf{p} - 3\mathbf{q}| \leqq 36$. Nothing about the direction.
32. $\mathbf{p} = -3\mathbf{i}, \mathbf{q} = -9\mathbf{j}, \mathbf{u} = 3\mathbf{k}$. Yes
34. TEAM PROJECT. (a) The idea is to write the position vector of P in the figure in two ways and then to compare,

$$\lambda(\mathbf{a} + \mathbf{b}) = \mathbf{a} + \mu(\mathbf{b} - \mathbf{a}).$$

$\lambda = 1 - \mu$ are the coefficients of **a** and $\lambda = \mu$ those of **b**. Together, $\lambda = \mu = \frac{1}{2}$, expressing bisection.

(b) The idea is similar to that in part (a). It gives

$$\lambda(\mathbf{a} + \mathbf{b}) = \tfrac{1}{2}\mathbf{a} + \mu\tfrac{1}{2}(\mathbf{b} - \mathbf{a}).$$

$\lambda = \frac{1}{2} - \frac{1}{2}\mu$ from **a** and $\lambda = \frac{1}{2}\mu$ from **b**, resulting in $\lambda = \frac{1}{4}$, thus a ratio 3:1.

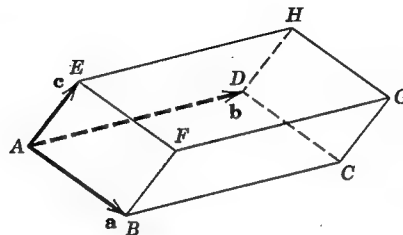
(c) Partition the parallelogram into four congruent parallelograms. Part (a) gives 1:1 for a small parallelogram, hence $1:(1 + 2)$ for the large parallelogram.

(d) In the figure, $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$, hence $\mathbf{c} + \mathbf{d} = -(\mathbf{a} + \mathbf{b})$. Also, $AB = \frac{1}{2}(\mathbf{a} + \mathbf{b})$, $CD = \frac{1}{2}(\mathbf{c} + \mathbf{d}) = -\frac{1}{2}(\mathbf{a} + \mathbf{b})$, and for DC we get $+\frac{1}{2}(\mathbf{a} + \mathbf{b})$, which shows that one pair of sides is parallel and of the same length. Similarly for the other pair.

(e) Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be the vectors. Their angle is $\alpha = 2\pi/n$. The interior angle at each vertex is $\beta = \pi - (2\pi/n)$. Put $\mathbf{v}_2$ at the terminal point of $\mathbf{v}_1$, then $\mathbf{v}_3$ at the terminal point of $\mathbf{v}_2$, etc. Then the figure thus obtained is an n-sided regular polygon, because the angle between two sides equals $\pi - \alpha = \beta$. Hence $\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n = \mathbf{0}$. (Of course, for *even n* the truth of the statement is immediately obvious).

(f) Let **a**, **b**, **c** be edge vectors with a common initial point (see the figure). Then the four (space) diagonals have the midpoints

$$AG:\ \tfrac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$$
$$BH:\ \mathbf{a} + \tfrac{1}{2}(\mathbf{b} + \mathbf{c} - \mathbf{a})$$
$$EC:\ \mathbf{c} + \tfrac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$$
$$DF:\ \mathbf{b} + \tfrac{1}{2}(\mathbf{a} + \mathbf{c} - \mathbf{b}),$$

and these four position vectors are equal.

Section 8.1. Parallelepiped in Team Project 34(f)

SECTION 8.2. Inner Product (Dot Product), page 408

Purpose. We define, explain, and apply a first kind of product of vectors, the dot product $\mathbf{a} \cdot \mathbf{b}$, whose value is a scalar.

Main Content, Important Concepts

Definition (1)

Dot product in terms of components

Orthogonality

Length and angle between vectors in terms of dot products

Schwarz and triangle inequalities

Comment on Dot Product

This product is motivated by work done by a force (Example 2), by the calculation of components of forces (Example 3), and by geometric applications such as those given in Examples 5 and 6.

"Inner product" is more modern than "dot product" and is also used in more general settings (see Sec. 6.8).

SOLUTIONS TO PROBLEM SET 8.2, page 413

2. $\sqrt{14}, 2\sqrt{14}, \sqrt{21}$

4. -16

6. -37

8. $\sqrt{27}, \sqrt{14} + \sqrt{29}$

10. $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$. $\mathbf{v} - \mathbf{w}$ is orthogonal to $\mathbf{u}$. So this does *not* imply that $\mathbf{v} - \mathbf{w} = \mathbf{0}$, that is, $\mathbf{v} = \mathbf{w}$.

14. 0

16. -82

18. Yes, because $W = (\mathbf{p} + \mathbf{q}) \cdot \mathbf{d} = \mathbf{p} \cdot \mathbf{d} + \mathbf{q} \cdot \mathbf{d}$.

20. $\dfrac{(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}||\mathbf{c} - \mathbf{a}|} = \dfrac{[2, 1, 1] \cdot [0, -1, 2]}{\sqrt{6}\sqrt{5}} = \dfrac{1}{\sqrt{30}}$. *Answer:* 79.5°

22. $\dfrac{(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}}{|\mathbf{a} + \mathbf{b}|\,|\mathbf{c}|} = \dfrac{[4, 3, 1] \cdot [1, 0, 2]}{\sqrt{26}\sqrt{5}} = \dfrac{6}{\sqrt{130}}$. *Answer:* 58.25°

24. 85.74°

26. 60°

28. 55.3°, 85.6°, 39.1°

30. 3

32. $2/\sqrt{11}$

34. 0

36. $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \leqq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 = (|\mathbf{a}| + |\mathbf{b}|)^2$

38. PROJECT. (a) $a_1 = 1$

(b) $\mathbf{a}$ such that $5a_1 = 2a_2$ with $a_1{}^2 + a_2{}^2 = 1$

(c) $\mathbf{b}$ such that $2b_1 + b_2 = 0$ and b_3 arbitrary. Yes

(d) $c = 3/4$

(e) $c = -2$

(f) $\mathbf{a} = [0, 0, 1]$ is a unit vector orthogonal to $\mathbf{b}$ and $\mathbf{c}$, and $q_1 = q_2 = 1/5$ gives unit vectors $\mathbf{b}$ and $\mathbf{c}$, which are orthogonal.

(g) If $\mathbf{a}$ and $\mathbf{b}$ correspond to adjacent sides, to the diagonals there correspond $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$. Orthogonality implies that

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2 = 0, \quad |\mathbf{a}| = |\mathbf{b}|.$$

SECTION 8.3. Vector Product (Cross Product), page 414

Purpose. We define and explain a second kind of product of vectors, the cross product $\mathbf{a} \times \mathbf{b}$, which is a vector perpendicular to both given vectors (or the zero vector in some cases).

Main Content, Important Concepts

Definition of cross product, its components (2), (2**)

Right- and left-handed coordinate systems

Properties (anticommutative, not associative)

Scalar triple product

Prerequisites. Elementary use of second- and third-order determinants (see the beginning of Sec. 6.6).

Comment on Motivations

Cross products were suggested by the observation that in certain applications, one associates with two given vectors a third vector perpendicular to the given vectors (illustrations in Examples 4–6). Scalar triple products can be motivated by volumes (Example 7) and linear independence (Theorem 1).

SOLUTIONS TO PROBLEM SET 8.3, page 421

2. $[8, 12, -13]$, $[-8, -12, 13]$, 0, 0

4. $[8, 12, 18]$

6. $\mathbf{0}$, $\mathbf{0}$, 13

8. $[4, -2, -27]$

10. $\mathbf{0}$

12. $[-16, -24, 0]$, $\mathbf{0}$

14. -62, -62

16. 32, 32

18. 1776

20. $\mathbf{v} = [3, 0, 0] \times [2, 2, 2] = [0, -6, 6]$, $|\mathbf{v}| = \sqrt{72}$

22. $\mathbf{m} = [-3, 4, 0] \times [2, 1, 0] = [0, 0, -11]$

24. $\mathbf{m} = [-3, -1, 2] \times [3, -1, 2] = [0, 12, 6]$

26. The area is obtained as the length of the cross product of two edge vectors,

$$[4 - 1, \quad -2 - 1, \quad 0] \times [9 - 1, \quad 3 - 1, \quad 0] = [0, \quad 0, \quad 30].$$

Answer: 30.

28. $[4 - 2, \quad -1 - 1, \quad 0] \times [6 - 2, \quad 3 - 1, \quad 0] = [0, \quad 0, \quad 12]$. *Answer:* 6

30. $[2 - 1, \quad 0 - 3, \quad 8 - 0] \times [0 - 1, \quad 2 - 3, \quad 2 - 0] = [2, -10, -4]$. Hence $2x - 10y - 4z = c$ with $c = -28$ obtained by inserting one of the three points.

32. 10

34. Edge vectors are $[3 - 1, \quad 7 - 3, \quad 12 - 6]$, $[8 - 1, \quad 8 - 3, \quad 9 - 6]$, $[2 - 1, \quad 2 - 3, \quad 8 - 6]$. The mixed triple product of these vectors is -90 (or $+90$). *Answer:* 15.

36. Their determinant is 220. *Answer:* Yes.

38. TEAM PROJECT. (a) $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2 \gamma = |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2 \gamma) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2$

(b) We choose a right-handed Cartesian coordinate system such that the x-axis has the direction of $\mathbf{d}$ and the xy-plane contains $\mathbf{c}$. Then the vectors in (b) are of the

form

$$\mathbf{b} = [b_1, b_2, b_3], \qquad \mathbf{c} = [c_1, c_2, 0], \qquad \mathbf{d} = [d_1, 0, 0].$$

Hence by (2**),

$$\mathbf{c} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & 0 \\ d_1 & 0 & 0 \end{vmatrix} = -c_2 d_1 \mathbf{k}, \quad \mathbf{b} \times (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ 0 & 0 & -c_2 d_1 \end{vmatrix}.$$

The determinant on the right equals $[-b_2c_2d_1, \quad b_1c_2d_1, \quad 0]$. Also,

$$\begin{aligned}(\mathbf{b}\bullet\mathbf{d})\mathbf{c} - (\mathbf{b}\bullet\mathbf{c})\mathbf{d} &= b_1d_1[c_1, c_2, 0] - (b_1c_1 + b_2c_2)[d_1, 0, 0] \\ &= [-b_2c_2d_1, \quad b_1d_1c_2, \quad 0].\end{aligned}$$

This proves (b) for our special coordinate system. Now the length and direction of a vector and a vector product, and the value of an inner product, are independent of the choice of the coordinates. Furthermore, the representation of $\mathbf{b} \times (\mathbf{c} \times \mathbf{d})$ in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ will be the same for right-handed and left-handed systems, because of the double cross multiplication. Hence, (b) holds in any Cartesian coordinate system, and the proof is complete.

(c) This follows from (b) with $\mathbf{b}$ replaced by $\mathbf{a} \times \mathbf{b}$.

(d) $\mathbf{a}\bullet[\mathbf{b} \times (\mathbf{c} \times \mathbf{d})]$ equals $(\mathbf{a} \quad \mathbf{b} \quad [\mathbf{c} \times \mathbf{d}]) = (\mathbf{a} \times \mathbf{b})\bullet(\mathbf{c} \times \mathbf{d})$ by the definition of the triple product, as well as $(\mathbf{a}\bullet\mathbf{c})(\mathbf{b}\bullet\mathbf{d}) - (\mathbf{a}\bullet\mathbf{d})(\mathbf{b}\bullet\mathbf{c})$ by (b) (take the dot product by $\mathbf{a}$).

SECTION 8.4. Vector and Scalar Functions and Fields. Derivatives, page 423

Purpose. To get started on vector differential calculus, we discuss vector functions and their continuity and differentiability.

Main Content, Important Concepts

Vector and scalar functions and fields

Continuity, derivative of vector functions (9), (10)

Differentiation of dot, cross, and triple products, (11)–(13)

Partial derivatives

Comment on Content

This parallels calculus of functions of one variable and, if known to students, can be surveyed quickly.

SOLUTIONS TO PROBLEM SET 8.4, page 427

2. Ellipses **4.** Circles **6.** Hyperbolas

8. Circles $\left(x - \dfrac{1}{2c}\right)^2 + y^2 = \dfrac{1}{4c^2}$

10. CAS PROJECT. Note that all these functions occur in connection with solutions of Laplace's equation; so they are real or imaginary parts of complex analytic functions.

For example, (f) occurs in connection with ln cos z. A CAS can graphically handle these more complicated functions, whereas the paper and pencil method is relatively limited. This is the point of the project.

12. Elliptic cylinders

14. Paraboloids of revolution

16. Congruent cylinders whose cross section in the yz-plane is a quadratic parabola

18. Note that each field vector equals the position vector of the corresponding point.

20. Note that each field vector is orthogonal to the position vector of the corresponding point, as for the velocity field of a rotation.

26. $-yz \sin xyz\,(\mathbf{i} + \mathbf{j}),\ -xz \sin xyz\,(\mathbf{i} + \mathbf{j}),\ -xy \sin xyz\,(\mathbf{i} + \mathbf{j})$

28. $[e^x \cos y,\ \ e^x \sin y,\ \ 0],\ [-e^x \sin y,\ \ e^x \cos y,\ \ 0]$

30. $\left[\dfrac{x}{x^2+y^2},\ \dfrac{-y}{x^2+y^2}\right],\ \left[\dfrac{y}{x^2+y^2},\ \dfrac{x}{x^2+y^2}\right]$

SECTION 8.5. Curves. Tangents. Arc Length, page 428

Purpose. Discussion of space curves as an application of vector functions of one variable and in view of later use of curves in mechanics (Sec. 8.6) and in line integrals (Chap. 9).

Main Content, Important Concepts

Parametric representation of curves (1)

Tangent vector, tangent, (7)–(9)

Arc length s

Comment on Problems 27–32

These involve only integrals that are simple (which is usually not the case in connection with lengths of curves).

SOLUTIONS TO PROBLEM SET 8.5, page 433

2. $\mathbf{r}(t) = [-1 + 3t,\ \ 3 + t,\ \ 8]$

4. $\mathbf{r}(t) = [1 - t,\ \ 1 + t,\ \ 1 - t]$

6. $\mathbf{r}(t) = [4t,\ \ 4t,\ \ t]$

8. $\mathbf{r}(t) = [a + 4t,\ \ b + (2 - 2b)t,\ \ c - t]$

10. Helix on an elliptical cylinder

12. Circle in the xy-plane with center at $(a, b, 0)$

14. Ellipse in the plane $z = 4$

16. Only the portion corresponding to $x \geqq 0$

18. Hyperbola $[\sqrt{3} \cosh t,\ \ 2 \sinh t,\ \ 1]$

20. Helix $[3 \cos t,\ \ 3 \sin t,\ \ 5t]$

22. (a) $\mathbf{r}'(t) = [-2 \sin t,\ \ 2 \cos t,\ \ 0],\ \mathbf{u} = \frac{1}{2}\mathbf{r}'$

(b) $\mathbf{r}'(P) = [-\sqrt{2},\ \ \sqrt{2},\ \ 0],\ \mathbf{u}(P) = [-1/\sqrt{2},\ \ 1/\sqrt{2},\ \ 0]$

(c) $\mathbf{q}(w) = [\sqrt{2} - \sqrt{2}w,\ \ 2 + \sqrt{2}w,\ \ 0]$

24. (a) $\mathbf{r}'(t) = [-2 \sin t,\ \ 2 \cos t,\ \ 1],\ \mathbf{u} = (1/\sqrt{5})\mathbf{r}'(t)$

(b) $\mathbf{r}'(P) = [0,\ \ 2,\ \ 1],\ \mathbf{u}(P) = [0,\ \ 2/\sqrt{5},\ \ 1/\sqrt{5}]$

(c) $\mathbf{q}(w) = [2,\ \ 2w,\ \ w]$

26. (a) $\mathbf{u} = (1 + 4t^2 + 9t^4)^{-1/2}(\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})$
(b) $\mathbf{u}(P) = 14^{-1/2}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$
(c) $(1 + w)\mathbf{i} + (1 + 2w)\mathbf{j} + (1 + 3w)\mathbf{k}$

28. $2\pi\sqrt{a^2 + c^2}$

30. $\mathbf{r}' = -3a\cos^2 t \sin t\,\mathbf{i} + 3a\sin^2 t\cos t\,\mathbf{j}$

$$\mathbf{r}'\bullet\mathbf{r}' = 9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t = 9a^2\cos^2 t\sin^2 t = \frac{9a^2}{4}\sin^2 2t.$$

This gives as the length in the first quadrant

$$l = \frac{3a}{2}\int_0^{\pi/2}\sin 2t\,dt = -\frac{3a}{4}(\cos\pi - \cos 0) = \frac{3a}{2}.$$

Answer: $6a$.

32. From the given representation we get $d\rho = a\sin\theta\,d\theta$. Hence

$$ds^2 = a^2(1 - \cos\theta)^2\,d\theta^2 + a^2\sin^2\theta\,d\theta^2 = 2a^2(1 - \cos\theta)\,d\theta^2$$

where $1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$, so that the total length is

$$l = 2a\int_0^{2\pi}\sin\tfrac{1}{2}\theta\,d\theta = -4a(\cos\pi - 1) = 8a.$$

SECTION 8.6. Velocity and Acceleration, page 435

Purpose. To show the role of parametric representations and of derivatives in connection with motions in mechanics.

Main Content, Important Concepts

Velocity vector

Acceleration vector, its tangent and normal components

Angular speed

Centripetal acceleration

Coriolis acceleration

Short Courses. This section (and the next two sections) can be omitted.

SOLUTIONS TO PROBLEM SET 8.6, page 439

2. $\mathbf{v} = \cos t\,\mathbf{i}$, $\mathbf{a} = -\sin t\,\mathbf{i} = \mathbf{a}_{tang}$, $\mathbf{a}_{norm} = \mathbf{0}$

4. $\mathbf{v} = -4\sin 2t\,\mathbf{i} - 4\cos 2t\,\mathbf{j}$, $\mathbf{a} = -8\cos 2t\,\mathbf{i} + 8\sin 2t\,\mathbf{j}$, $\mathbf{a}_{tang} = \mathbf{0}$, $\mathbf{a}_{norm} = \mathbf{a}$, as in Example 1.

6. $\mathbf{v} = -\sin t\,\mathbf{i} + 2\cos 2t\,\mathbf{j}$, $\mathbf{a} = -\cos t\,\mathbf{i} - 4\sin 2t\,\mathbf{j}$,

$$\mathbf{a}_{tang} = \frac{\mathbf{a}\bullet\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\frac{1}{2}\sin 2t - 4\sin 4t}{\sin^2 t + 4\cos^2 2t}\mathbf{v}, \qquad \mathbf{a}_{norm} = \mathbf{a} - \mathbf{a}_{tang}.$$

The path looks like an infinity sign, with a double point at the origin, corresponding to $t = \pi/2$ and $3\pi/2$.

8. CAS PROJECT. (a) $\mathbf{v} = [-2\sin t - 2\sin 2t, \quad 2\cos t - 2\cos 2t]$. From this we obtain $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v} = (-2\sin t - 2\sin 2t)^2 + (2\cos t - 2\cos 2t)^2$. Performing the squares and simplifying gives

$$\begin{aligned} |\mathbf{v}|^2 &= 8(1 + \sin t \sin 2t - \cos t \cos 2t) \\ &= 8(1 - \cos 3t) \\ &= 16\sin^2\frac{3t}{2}. \end{aligned}$$

Hence

$$|\mathbf{v}| = 4\sin\frac{3t}{2}.$$

$$\mathbf{a} = [-2\cos t - 4\cos 2t, \quad -2\sin t + 4\sin 2t].$$

We use (4*). By straightforward simplification,

$$\begin{aligned} \mathbf{a}\cdot\mathbf{v} &= 12(\cos t \sin 2t + \sin t \cos 2t) \\ &= 12\sin 3t. \end{aligned}$$

Hence (4*) gives

$$\mathbf{a}_{\text{tan}} = \frac{12\sin 3t}{16\sin^2(3t/2)}\mathbf{v}$$

$$\mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\text{tan}}.$$

(b) $\mathbf{v} = [-\sin t - 2\sin 2t, \quad \cos t - 2\cos 2t]$

$$|\mathbf{v}|^2 = 5 - 4\cos 3t$$

$$\mathbf{a} = [-\cos t - 4\cos 2t, \quad -\sin t + 4\sin 2t]$$

$$\mathbf{a}_{\text{tan}} = \frac{6\sin 3t}{5 - 4\cos 3t}[-\sin t - 2\sin 2t, \quad \cos t - 2\cos 2t]$$

$$\mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\text{tan}}$$

(c) $\mathbf{v} = [-\sin t, 2\cos 2t, \quad -2\sin 2t]$

$$|\mathbf{v}|^2 = 4 + \sin^2 t$$

$$\mathbf{a} = [-\cos t, \quad -4\sin 2t, \quad -4\cos 2t]$$

$$\mathbf{a}_{\text{tan}} = \frac{\frac{1}{2}\sin 2t}{4 + \sin^2 t}[-\sin t, \quad 2\cos 2t, \quad -2\sin 2t]$$

$$\mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\text{tan}}$$

(d) $\mathbf{v} = [c\cos t - ct\sin t, \quad c\sin t + ct\cos t, \quad c]$

$$|\mathbf{v}|^2 = c^2(t^2 + 2)$$

$$\mathbf{a} = [-2c\sin t - ct\cos t, \quad 2c\cos t - ct\sin t, \quad 0]$$

$$\mathbf{a}_{\text{tan}} = \frac{ct}{t^2 + 2}[\cos t - t\sin t, \quad \sin t + t\cos t, \quad 1]$$

$$\mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\text{tan}}$$

This is a spiral on a cone.

10. $\mathbf{v} = [-\sin t, \quad 2\cos t]$, $|\mathbf{v}|^2 = 4\cos^2 t + \sin^2 t$ is minimum at $\pm\pi/2$ and maximum at 0 and π. Also, $\mathbf{a} = [-\cos t, \quad -2\sin t]$, and $|\mathbf{a}|$ is minimum at $t = 0$ and π (on the x-axis) and maximum at $\pm\pi/2$ (on the y-axis). Finally,

$$\mathbf{a}_{\text{tan}} = \frac{3\sin t\cos t}{\sin^2 t + 4\cos^2 t}[\sin t, \quad -2\cos t]$$

$$\mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\text{tan}}$$

12. $\mathbf{r} = t^2\mathbf{b}$, $\mathbf{v} = 2t\mathbf{b} + t^2\mathbf{b}'$, $\mathbf{a} = \mathbf{v}' = 2\mathbf{b} + 4t\mathbf{b}' + t^2\mathbf{b}''$. *Answer:* $4t\mathbf{b}'$

14. $R = 3.85 \cdot 10^8$ m, $\quad |\mathbf{v}| = 2\pi R/(2.36 \cdot 10^6) = 1025$ [m/sec], $|\mathbf{v}| = \omega R$, $|\mathbf{a}| = \omega^2 R = |\mathbf{v}|^2/R = 0.0027$ [meter/sec^2], which is only $2.8 \cdot 10^{-4} g$, where g is the acceleration due to gravity at the earth's surface.

16. $R = 3960 + 450 = 4410$ [mi], $2\pi R = 100|\mathbf{v}|$, $|\mathbf{v}| = 277.1$ mi/min, $g = |\mathbf{a}| = \omega^2 R = |\mathbf{v}|^2/R = 17.41$ [mi/min^2] $= 25.53$ [ft/sec^2] $= 7.78$ [meters/sec^2]. Here we used $|\mathbf{v}| = \omega R$.

SECTION 8.7. Curvature and Torsion of a Curve. *Optional,* page 440

Purpose. To complete the discussion of the foundations of differential geometric curve theory. We leave this section ***optional*** because we shall not refer to curvature or torsion in our further work.

Main Content, Important Concepts

Curvature

Torsion

Frenet formulas

Short Courses. Omit this section.

SOLUTIONS TO PROBLEM SET 8.7, page 443

2. We denote derivatives with respect to t by primes. In (1),

$$\mathbf{u} = \frac{d\mathbf{r}}{ds} = \mathbf{r}'\frac{dt}{ds}, \qquad \frac{dt}{ds} = \frac{1}{s'} = (\mathbf{r}'\bullet\mathbf{r}')^{-1/2}. \quad \text{(See (12), Sec. 8.5.)}$$

Thus in (1),

$$\frac{d\mathbf{u}}{ds} = \mathbf{r}''\left(\frac{dt}{ds}\right)^2 + \mathbf{r}'\frac{d^2t}{ds^2} = \mathbf{r}''(\mathbf{r}'\bullet\mathbf{r}')^{-1} + \mathbf{r}'\frac{d^2t}{ds^2}$$

where

$$\frac{d^2t}{ds^2} = \frac{d}{dt}\left(\frac{dt}{ds}\right)\frac{dt}{ds} = -\tfrac{1}{2}(\mathbf{r}'\bullet\mathbf{r}')^{-3/2}\,2(\mathbf{r}''\bullet\mathbf{r}')(\mathbf{r}'\bullet\mathbf{r}')^{-1/2}$$

$$= -(\mathbf{r}''\bullet\mathbf{r}')(\mathbf{r}'\bullet\mathbf{r}')^{-2}.$$

Hence

$$\frac{d\mathbf{u}}{ds} = \mathbf{r}''(\mathbf{r}'\bullet\mathbf{r}')^{-1} - \mathbf{r}'(\mathbf{r}''\bullet\mathbf{r}')(\mathbf{r}'\bullet\mathbf{r}')^{-2}$$

$$\frac{d\mathbf{u}}{ds}\bullet\frac{d\mathbf{u}}{ds} = (\mathbf{r}''\bullet\mathbf{r}'')(\mathbf{r}'\bullet\mathbf{r}')^{-2} - 2(\mathbf{r}''\bullet\mathbf{r}')^2(\mathbf{r}'\bullet\mathbf{r}')^{-3} + (\mathbf{r}'\bullet\mathbf{r}')^{-3}(\mathbf{r}''\bullet\mathbf{r}')^2$$
$$= (\mathbf{r}''\bullet\mathbf{r}'')(\mathbf{r}'\bullet\mathbf{r}')^{-2} - (\mathbf{r}''\bullet\mathbf{r}')^2(\mathbf{r}'\bullet\mathbf{r}')^{-3}.$$

Taking square roots, we get (1′).

4. Ellipse $x^2/a^2 + y^2/b^2 = 1, \quad \kappa = ab(a^2 \sin^2 t + b^2 \cos^2 t)^{-3/2}$

6. Hyperbola, $\kappa = 2|cx^3|/(x^4 + c^2)^{3/2}$

8. Hyperbola $x^2 - y^2 = 1$, $(\cosh^2 t + \sinh^2 t)^{-3/2}$

10. $\tau = -\mathbf{p}\bullet(\mathbf{u}\times\mathbf{p})' = -\mathbf{p}\bullet(\mathbf{u}'\times\mathbf{p} + \mathbf{u}\times\mathbf{p}') = 0 - (\mathbf{p}\quad\mathbf{u}\quad\mathbf{p}') = +(\mathbf{u}\quad\mathbf{p}\quad\mathbf{p}')$

12. $\dfrac{d\mathbf{r}}{ds} = \dfrac{d\mathbf{r}}{dt}\Big/\dfrac{ds}{dt}, \quad \dfrac{d^2\mathbf{r}}{ds^2} = \dfrac{d^2\mathbf{r}}{dt^2}\Big/\left(\dfrac{ds}{dt}\right)^2 + \cdots, \quad \dfrac{d^3\mathbf{r}}{ds^3} = \dfrac{d^3\mathbf{r}}{dt^3}\Big/\left(\dfrac{ds}{dt}\right)^3 + \cdots,$
where the dots denote terms that vanish by applying familiar rules for simplifying determinants; thus

$$\tau = \frac{1}{\kappa^2}\left(\frac{d\mathbf{r}}{ds}\quad\frac{d^2\mathbf{r}}{ds^2}\quad\frac{d^3\mathbf{r}}{ds^3}\right) = \frac{1}{\kappa^2(ds/dt)^6}\left(\frac{d\mathbf{r}}{dt}\quad\frac{d^2\mathbf{r}}{dt^2}\quad\frac{d^3\mathbf{r}}{dt^3}\right).$$

Now use (1′) and formula (12) in Sec. 8.5.

14. $c/(a^2 + c^2)$

16. $\mathbf{p} = \mathbf{b}\times\mathbf{u}, \mathbf{p}' = \mathbf{b}'\times\mathbf{u} + \mathbf{b}\times\mathbf{u}' = -\tau\mathbf{p}\times\mathbf{u} + \mathbf{b}\times\kappa\mathbf{p}$
$= -\tau(-\mathbf{b}) + \kappa(-\mathbf{u})$

SECTION 8.8. Review from Calculus in Several Variables. *Optional,* page 443

Purpose. To give the student a handy reference and some help on material known from calculus, if needed.

SOLUTIONS TO PROBLEM SET 8.8, page 446

2. $g'/h - gh'/h^2$

4. $(1 - t - \sin t)\sin t + (1 + t + \cos t)\cos t$

6. $4u, 4v$

8. $4u^3(v^4 - 4 + v^{-4}), 4u^4(v^3 - v^{-5})$

10. This follows from (1). *Answer:* $3x^2 + 2(x^2 + y^2)2x, 3y^2 + 2(x^2 + y^2)2y$.

SECTION 8.9. Gradient of a Scalar Field. Directional Derivative, page 446

Purpose. To discuss gradients and their role in connection with directional derivatives, surface normals, and the generation of vector fields from scalar fields (potentials).

Main Content, Important Concepts

Gradient, nabla operator

Directional derivative, maximum increase, surface normal

Vector fields as gradients of potentials

Laplace's equation

Comments on Content

This is probably the first section in which one should no longer rely on knowledge from calculus, although relatively elementary calculus books usually include a passage on gradients.

Potentials are important; they will occur at a number of places in our further work.

SOLUTIONS TO PROBLEM SET 8.9, page 452

2. $[y, x]$, $[1, 1]$

4. $[2x, \quad 18y]$, $[-4, \quad 36]$

6. $[e^x \sin y, e^x \cos y]$, $[\sqrt{2}, \sqrt{2}]$

8. $-(\cos(x + z))\,[1, \quad 0, \quad 1]$, $\left[-\dfrac{1}{\sqrt{2}}, 0, -\dfrac{1}{\sqrt{2}}\right]$

10. $(x^2 + y^2)^{-1}[y, \quad -x]$, $[\frac{4}{25}, \quad -\frac{3}{25}]$

12. $-2e^{x^2-y^2}\,[(x \sin 2xy + y \cos 2xy)\mathbf{i} + (x \cos 2xy - y \sin 2xy)\mathbf{j}]$,
$-2(\sin 2 + \cos 2)\mathbf{i} - 2(\cos 2 - \sin 2)\mathbf{j}$

14. The direction of $6\mathbf{i} - 5\mathbf{j}$

16. $(1/\sqrt{5})[2, 1]$

18. $(a^2 + b^2 + c^2)^{-1/2}\,[a, b, c]$

20. $(1/\sqrt{11})[1, 1, 3]$

22. xyz

24. $ye^x + z$

26. $\frac{1}{2} \ln (x^2 + y^2)$

28. PROJECT. The first formula follows from

$$[(fg)_x, (fg)_y, (fg)_z] = [f_x g, f_y g, f_z g] + [f g_x, f g_y, f g_z].$$

The second formula follows by the chain rule, and the third follows by applying the quotient rule to each of the components $(f/g)_x$, $(f/g)_y$, $(f/g)_z$ and suitably collecting terms. The last formula follows by two applications of the product rule to each of the three terms of ∇^2.

30. $1/\sqrt{5}$

32. $1/(2\sqrt{2})$

34. $2\sqrt{3}$

SECTION 8.10. Divergence of a Vector Field, page 453

Purpose. To explain the divergence (the second of the three concepts grad, div, curl) and its physical meaning in fluid flows.

Main Content, Important Concepts

Divergence of a vector field

Continuity equations (5), (6)

Incompressibility condition (7)

Comment on Content

The interpretation of the divergence in Example 2 depends essentially on our assumption that there are no sources or sinks in the box. From our calculations it becomes plausible that in the case of sources or sinks the divergence may be related to the net flow across the boundary surfaces of the box. To confirm this and to make it precise we need integrals; we shall do this in Sec. 9.8 (in connection with Gauss's divergence theorem).

Moving div and curl to Chap. 9?

Experimentation has shown that this would perhaps not be a good idea, simply because it would combine two substantial difficulties, that of understanding div and curl themselves, and that of understanding the nature and role of the two basic integral theorems by Gauss and Stokes, in which div and curl play the key role.

SOLUTIONS TO PROBLEM SET 8.10, page 456

2. $2x + 2y + 2z$ **4.** 0

6. 0 **8.** $6xyz$

10. $\mathbf{w} \times \mathbf{r} = (w_2z - w_3y)\mathbf{i} + (w_3x - w_1z)\mathbf{j} + (w_1y - w_2x)\mathbf{k}$ shows immediately that div $\mathbf{v} = 0$ because the first, second, and third components do not depend on x, y, z, respectively.

12. $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = x\mathbf{i}$. Hence div $\mathbf{v} = 1$, and

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = 0, \qquad \frac{dz}{dt} = 0.$$

By integration, $x = c_1e^t$, $y = c_2$, $z = c_3$, and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Hence

$$\mathbf{r}(0) = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \qquad \text{and} \qquad \mathbf{r}(1) = c_1e\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

This shows that the cube in Prob. 11 is transformed into the rectangular parallelepiped bounded by $x = 0, x = e, y = 0, y = 1, z = 0, z = 1$, whose volume is e.

14. $4(x - y)/(x + y)^3$

16. 0

18. $2 \cosh 2x - 2 \cosh 2y$

20. $-1/\sqrt{x^2 + y^2}$. Equation (3) is simpler than differentiation.

SECTION 8.11. Curl of a Vector Field, page 457

Purpose, Content

We introduce the curl of a vector field (the last of the three concepts grad, div, curl) and interpret it in connection with rotations [Example 2 and the remarks on (3) and (4)]. A main application of the curl follows in Sec. 9.9 in Stokes's integral theorem.

SOLUTIONS TO PROBLEM SET 8.11, page 459

2. $3\mathbf{k}$ **4. 0**

6. $[\sin z, \quad 0, \quad -\cos y]$

8. curl $\mathbf{v} = -4y\mathbf{k}$, div $\mathbf{v} = 0$, imcompressible, $\mathbf{v} = [x', y', z'] = [2y^2, 0, 0]$, $y' = 0, y = c_2, z' = 0, z = c_3, x' = 2y^2 = 2c_2^{\,2}, x = 2c_2^{\,2}t + c_1$

10. curl $\mathbf{v} = [0, 0, (\operatorname{cosec} x)']$, div $\mathbf{v} = \sec x \tan x$, $\mathbf{v} = [x', y', z'] = [\sec x, \operatorname{cosec} x, 0]$, $x' = \sec x$, $\cos x\, dx = dt$, $\sin x = t + c_1$, $x = \arcsin(t + c_1)$, $y' = \operatorname{cosec} x = 1/(t + c_1)$, $y = \ln(t + c_1) + c_2$, $z = c_3$

12. curl $\mathbf{v} = (17/4)\mathbf{k}$, incompressible, $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k} = -\frac{1}{4}y\mathbf{i} + 4x\mathbf{j}$; hence (a) $x' = -\frac{1}{4}y$, (b) $y' = 4x$. Now (b) gives $x = \frac{1}{4}y'$. From this and (a) we obtain

$xx' = -(\frac{1}{4}y)(\frac{1}{4}y')$. By integration, $x^2/2 = -y^2/32 + const$, $x^2 + (y/4)^2 = const.$ The streamlines are ellipses.

14. PROJECT. Parts (b) and (d) are basic. They follow from the definitions by direct calculation. Part (a) follows by decomposing each component accordingly.

(c) In the first component in (1) we now have fv_3 instead of v_3, etc. Product differentiation gives $(fv_3)_y = f_y v_3 + f(v_3)_y$. Similarly for the other five terms in the components. $f_y v_3$ and the corresponding five terms give (grad f) $\times$ **v** and the other six terms $f(v_3)_y$, etc. give f curl **v**.

(d) For twice continuously differentiable f, for which the mixed second derivatives are equal, this follows from $\nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$ and (1), which gives

$$\text{curl}\,(\nabla f) = [(f_z)_y - (f_y)_z]\mathbf{i} + [(f_x)_z - (f_z)_x]\mathbf{j} + [(f_y)_x - (f_x)_y]\mathbf{k}.$$

(e) Write out and compare the twelve terms on either side.

16. **0**, 0

18. $[-xy + zx - z^2, \quad -yz + xy - x^2, \quad -zx + yz - y^2]$

20. $-x - y - z$, $[x - z, \quad y - x, \quad z - y]$

Solutions to Chapter 8 Review, page 461

14. -1716

16. $[16, -60, -6]$, 0

18. $[156, 104, -124]$, $[2, -6, 0]$

20. $(1/\sqrt{14})\,[3, 1, -2]$, $(1/\sqrt{52})\,[4, -6, 0]$

22. **0**, $[-46, 106, -16]$

24. $\sqrt{72} < \sqrt{14} + \sqrt{74}$

26. $\arccos(-8/\sqrt{14 \cdot 74}) = 104.4°$, 0 (orthogonal vectors)

28. $\mathbf{p} = [1, 2, -7]$

30. $\mathbf{a} \cdot \mathbf{b}/|\mathbf{b}| = 12/\sqrt{18} = 2\sqrt{2}$

32. $\mathbf{m} = \mathbf{r} \times \mathbf{p} = [-1, 2, 0] \times [3, 8, 0] = -14\mathbf{k}$. The minus sign indicates that the tendency of rotation is in the clockwise sense.

34. 2/3

36. $\mathbf{v} = [-3 \sin t, -2 \cos t, \frac{1}{2}] = [-3/\sqrt{2}, -\sqrt{2}, \frac{1}{2}]$, $|\mathbf{v}| = 3\sqrt{3}/2$,
$\mathbf{a} = [-3 \cos t, 2 \sin t, 0]$

38. $[2y^2(x - z), 2y(x - z)^2, -2y^2(x - z)]$, **0**

40. $[-2, -4, -2]$, $[2y, 6z, 4x]$

42. $2y(3x - z)$

44. $-4xy + 2y^2 + 2xz - yz - 2z^2$

46. 0

48. 0, 0

50. 0 because this is a scalar triple product corresponding to a determinant with two equal rows.

CHAPTER 9 Vector Integral Calculus. Integral Theorems

SECTION 9.1. Line Integrals, page 464

Purpose. To explain line integrals in space and in the plane conceptually and technically with regard to their evaluation by using the representation of the path of integration.

Main Content, Important Concepts

Line integral (3), (3′), its evaluation

Its motivation by work done by a force

General properties (8)

Dependence on path (Example 3)

Background Material. Parametric representation of curves (Sec. 8.5); a couple of review problems may be useful.

Comments on Content

The integral (3) is more practical than (7) (more direct in view of subsequent material), and work done by a force motivates it sufficiently well.

Independence of path is settled in the next section.

SOLUTIONS TO PROBLEM SET 9.1, page 470

2. 6/5
4. For instance, $\mathbf{r} = [2 - 2t,\ 2t]$, $0 \leqq t \leqq 1$. *Answer:* $-4/15$
6. For instance, $\mathbf{r} = [t,\ t^{3/2}]$. *Answer:* 0
8. $\mathbf{F}(C) = [2\cos t - t,\quad t - 2\sin t,\quad 2\sin t - 2\cos t]$. *Answer:* $-8\pi + 2\pi^2$
10. $\mathbf{F}(C) = [\cosh t,\quad \sinh(t^2),\quad \exp(t^3)]$, $\mathbf{F}(C)\cdot\mathbf{r}' =$ $\cosh t + 2t\sinh(t^2) + 3t^2\exp(t^3)$. *Answer:* $\sinh 2 + \cosh 4 + e^8 - 2 \approx 3010$
12. **PROJECT. (a)** For $t = p^2$ we obtain $\mathbf{r}' = [-2p\sin p^2,\quad 2p\cos p^2]$, $\mathbf{F}(C) = [-\cos^2 p^2,\quad \cos p^2 \sin p^2]$, so that the integrand is $4p\cos^2 p^2 \sin p^2$. Integration gives $-(2/3)\cos^3 p^2$, hence 4/3, in agreement with the result for the given representation.

    (b) $\mathbf{r}' = [1,\quad nt^{n-1}]$, $\mathbf{F}(C) = [-t^2,\quad t^{n+1}]$, the integrand is $-t^2 + nt^{2n}$. By integration,

    $$-\frac{1}{3} + \frac{n}{2n+1}.$$

    (c) The limit is $-\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$. Direct integration from (0,0) to (1,0) and then to (1,1) gives the same, where the two summands correspond to the horizontal and the vertical part of the path.

14. $ds/dt = \sqrt{\mathbf{r}'\cdot\mathbf{r}'} = 2$, $\displaystyle\int_0^{\pi/2} 32\cos^3 t \sin t\, dt = 8$
16. $ds/dt = \sqrt{2 + 4t^2}$, $\displaystyle\int_0^3 (2 + 4t^2)\, dt = 6 + 36 = 42$

18. $ds/dt = \cosh t, \quad \int_0^2 (1 - \sinh^2 t) \cosh t \, dt = \sinh 2 - \frac{1}{3}\sinh^3 2 \approx -12.276$

20. $ds/dt = \sqrt{9\sin^2 t + 4\cos^2 t}, \quad \int_0^{\pi} 6(9\sin^2 t + 4\cos^2 t)\, dt = 39\pi$

SECTION 9.2. Line Integrals Independent of Path, page 471

Purpose. Independence of path is a basic issue on line integrals, and we discuss it here in full.

Main Content, Important Concepts

Definition of independence of path

Relation to gradient (Theorem 1), potential theory

Integration around closed curves

Work, conservative systems

Relation to exactness of differential forms

Comment on Content

We see that our text pursues three ideas by relating path independence to (i) gradients (potentials), (ii) closed paths, and (iii) exactness of the form under the integral sign. The complete proof of the latter needs Stokes's theorem, so here we leave a small gap to be easily filled in Sec. 9.9.

SOLUTIONS TO PROBLEM SET 9.2, page 477

2. $f = e^x \cos y$. *Answer:* 1

4. $f = \sin x \cos 2y$. *Answer:* $1/\sqrt{2} - 1$

6. $f = 3xy + z^2$. *Answer:* 16

8. $f = \cos xz + \sin y$. *Answer:* -1

10. PROJECT. **(a)** $2y^2 \neq x^2$ from (6″).

(b) $\mathbf{r} = [t, \ bt]$, $0 \leqq t \leqq 1$, represents the first part of the path. By integration, $b/4 + b^3/2$. On the second part, $\mathbf{r} = [1, t]$, $b \leqq t \leqq 1$. Integration gives $2(1 - b^3)/3$. Equating the derivative of the sum of the two expressions to zero gives $b = 1/\sqrt{2}$. The corresponding maximum value of I is $1/(6\sqrt{2}) + 2/3 = 0.78452$.

(c) The first part is $y = x/c$ or $\mathbf{r} = [t, \ t/c]$, $0 \leqq t \leqq c$. The integral over this portion is $c^3/4 + c/2$. For the second portion $\mathbf{r} = [t, \ 1]$, $c \leqq t \leqq 1$ the integral is $(1 - c^3)/3$. For $c = 1$ we get $I = 0.75$, the same as in (b) for $b = 1$. This is the maximum value of I for the present paths through $(c, 1)$ because the derivative of I with respect to c is positive for $0 \leqq c \leqq 1$.

12. Dependent on path

14. Independent of path, $f = xyz$, abc

16. Dependent on path

18. Dependent on path

20. Independent of path, $f = z\cosh y - x^2$, $c\cosh b - a^2$

SECTION 9.3. From Calculus: Double Integrals. *Optional,* page 478

Purpose. We need double integrals (and line integrals) in the next section and review them here for completeness, suggesting that the student go on to the next section.

Comment

Definition, evaluation, and properties of double integrals

Some standard applications

Change of variables, Jacobians

SOLUTIONS TO PROBLEM SET 9.3, page 484

2. Integrate over y to get

$$\int_0^4 \left(\tfrac{8}{3} + 2x^2\right) dx = \tfrac{160}{3},$$

as before.

4. This order of integration is less practical because we have to split the integral into two parts,

$$\int_{-3}^{0}\int_{-x}^{3} (x^2 + y^2)\, dy\, dx + \int_0^3\int_x^3 (x^2 + y^2)\, dy\, dx.$$

By integration over y we get from the first part

$$\int_{-3}^{0} \left[x^2(3 + x) + \frac{27}{3} + \frac{x^3}{3}\right] dx = 27.$$

The other part gives 27, too. *Answer:* 54.

6. After the integration over y we have

$$\int_0^{\pi/4} \frac{x}{2}(\cos^2 x - \sin^2 x)\, dx = -\frac{1}{8} + \frac{\pi}{16}.$$

8. We now have

$$\int_0^2\int_x^2 \sinh(x + y)\, dy\, dx = \int_0^2 [\cosh(2 + x) - \cosh 2x]\, dx = \tfrac{1}{2}\sinh 4 - \sinh 2.$$

10. After the integration over x we have

$$\int_0^{\pi/4} \frac{1}{3}\cos^3 y \sin y\, dy = \frac{1}{16}.$$

12. $\displaystyle\int_0^1\int_0^{1-x^2}\int_0^{1-x^2} dy\, dz\, dx = \frac{8}{15}$

14. $2b/3$, $h/3$

16. $\displaystyle\frac{4}{\pi a^2}\int_0^{\pi/2}\int_0^a (r\cos\theta)\, r\, dr\, d\theta = \frac{4}{\pi a^2}\int_0^{\pi/2} \frac{a^3}{3}\cos\theta\, d\theta = \frac{4a}{3\pi}$

18. $I_x = bh^3/12$, $I_y = 7b^3h/48$

20. $I_x = h^3(3b + a)/12$, $I_y = h(a^4 - b^4)/(48(a - b))$

SECTION 9.4. Green's Theorem in the Plane, page 485

Purpose. To state, prove, and apply Green's theorem in the plane, relating line and double integrals.

Comment on the Role of Green's Theorem in the Plane

This theorem is a special case of each of the two "big" integral theorems in this chapter, Gauss's and Stokes's theorems (Secs. 9.7, 9.9), but we need it as the essential tool in the proof of Stokes's theorem.

The present theorem must not be confused with *Green's first and second theorems* in Sec. 9.8.

SOLUTIONS TO PROBLEM SET 9.4, page 490

2. 4

4. $\mathbf{F} = \operatorname{grad} x^2y^3$. *Answer:* 0

6. $\int_0^{\pi}\int_0^{x/\pi} (-\cos y - \sin x)\, dy\, dx = -1 - \pi + \pi \cos 1$

8. $\int_1^3\int_x^{3x} (-\cosh x - \sinh y)\, dy\, dx = \int_1^3 (\cosh x - 2x\cosh x - \cosh 3x)\, dx$

$= 2(\cosh 3 - \cosh 1) + \sinh 1 - \frac{1}{3}(14 \sinh 3 + \sinh 9)$

10. $\int_0^1\int_{x^2}^{x} 2x \sinh 2y\, dy\, dx = \frac{1}{4}(1 - \sinh 2 - \cosh 2 + 2 \sinh 2) = \frac{1}{4}(1 - e^{-2})$

12. $\mathbf{r}' = [a(1 - \cos t), \quad a \sin t]$,

$$A = -\tfrac{1}{2}a^2 \int_0^{2\pi} [(t - \sin t) \sin t - (1 - \cos t)^2]\, dt = 3\pi a^2.$$

Here, the minus sign was needed because the sense of integration was such that the region was to the right of the curve.

14. $e^2 + 2e - 3 \approx 9.83$

16. $\nabla^2 w = 0$. *Answer:* 0

18. $\nabla^2 w = 20(x^3y + xy^3) = 20xy(x^2 + y^2) = 20r^4 \cos\theta \sin\theta$, so that we obtain

$$\int_0^{\pi/2}\int_0^1 20r^4 \cos\theta \sin\theta\, r\, dr\, d\theta = \frac{20}{6}\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta = \frac{5}{3}.$$

20. PROJECT. We obtain $\operatorname{div} \mathbf{F}$ in (11) if we take $\mathbf{F} = [F_2, -F_1]$. Taking $\mathbf{n} = [y', -x']$ as in Example 4, we get from (1) the right side in (11),

$$(\mathbf{F} \cdot \mathbf{n})\, ds = \left(F_2 \frac{dy}{ds} + F_1 \frac{dx}{ds}\right) ds = F_2\, dy + F_1\, dx.$$

Formula (12) follows from the explanation of (1′).

Furthermore, $\operatorname{div} \mathbf{F} = 7 - 3 = 4$ times the area of the disk of radius 2 gives 16π. For the line integral in (11) we need

$$\mathbf{r} = \left[2\cos\frac{s}{2}, 2\sin\frac{s}{2}\right], \qquad \mathbf{r}' = \left[-\sin\frac{s}{2}, \cos\frac{s}{2}\right], \qquad \mathbf{n} = [y', -x'].$$

This gives

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C (7xy' + 3yx')\, ds = \int_0^{4\pi} \left(14 \cos^2 \frac{s}{2} - 6 \sin^2 \frac{s}{2}\right) ds = 16\pi.$$

In (12) we have curl $\mathbf{F} = \mathbf{0}$ and

$$\mathbf{F} \cdot \mathbf{r}' = -14 \cos \frac{s}{2} \sin \frac{s}{2} - 6 \cos \frac{s}{2} \sin \frac{s}{2} = -10 \sin s,$$

which gives zero upon integration from 0 to 4π.

SECTION 9.5. Surfaces for Surface Integrals, page 491

Purpose. The section heading indicates that we are dealing with a tool in surface integrals, and we concentrate our discussion accordingly.

Main Content, Important Concepts

Parametric surface representation (2) (see also Fig. 221)

Surface normal vector $\mathbf{N}$, unit surface normal vector $\mathbf{n}$

Short Courses. Discuss (2) and (4) and a simple example.

Comments on Text and Problems

The student should realize and understand that the present parametric representations are the two-dimensional analog of parametric curve representations.

Problems 1–10 concern some standard surfaces of interest in applications. We shall need only a few of these surfaces, but these problems should help students grasp the idea of a parametric representation and see the relation to representations (1). Moreover, it may be good to collect surfaces of practical interest in one place for possible reference.

SOLUTIONS TO PROBLEM SET 9.5, page 495

2. $u\mathbf{k}$; note that this normal vector becomes the zero vector at the origin, where (4) is violated.

4. $z = x^2 + y^2$, circles, parabolas, $[-2u^2 \cos v, \quad -2u^2 \sin v, \quad u]$

6. $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, ellipses,

$$[bc \cos^2 v \cos u, \qquad ac \cos^2 v \sin u, \qquad ab \cos v \sin v]$$

8. $x^2/a^2 + y^2/b^2 - z^2/c^2 + 1 = 0$, ellipses, hyperbolas,

$$[-bc \sinh^2 u \cos v, \qquad -ac \sinh^2 u \sin v, \qquad ab \cosh u \sinh u]$$

10. $z = \arctan (y/x)$, helices, horizontal straight lines, $[\sin v, \quad -\cos v, \quad u]$

12. $[u, v, u]$, $[-1, 0, 1]$

14. $[2 \cos u, \quad 3 \sin u, v]$, $[3 \cos u, \quad 2 \sin u, \quad 0]$

16. $[2 \cos v \cos u, \quad 1 + 2 \cos v \sin u, \quad -2 + 2 \sin v]$,

$$[4 \cos u \cos^2 v, \quad 4 \sin u \cos^2 v, \quad 4 \sin v \cos v]$$

18. $[2u \cos v, \quad u \sin v, \quad 2u]$, $[-2u \cos v, \quad -4u \sin v, \quad 2u]$

20. At the origin, in Prob. 3 because of the apex of the cone, in the other problems because of the representation.

22. Because $\mathbf{r}_u$ and $\mathbf{r}_v$ are tangent to the coordinate curves $v = const$ and $u = const$, respectively.

24. $(16x^2 + y^2 + 81z^2)^{-1/2}$ $[4x, y, 9z]$

26. $(1 + 25x^2 + 25y^2)^{-1/2}$ $[5y, 5x, -1]$

28. $(x^2 + y^2 + z^2)^{-1/2}$ $[x, -y, z]$

30. **PROJECT.** **(a)** $\mathbf{r}_u(P)$ and $\mathbf{r}_v(P)$ span $T(P)$. $\mathbf{r}^*$ varies over $T(P)$. The vanishing of the scalar triple product implies that $\mathbf{r}^* - \mathbf{r}(P)$ lies in the tangent plane $T(P)$.

   (b) Geometrically, the vanishing of the dot product means that $\mathbf{r}^* - \mathbf{r}(P)$ must be perpendicular to ∇g, which is a normal vector of S at P.

   (c) Geometrically, $f_x(P)$ and $f_y(P)$ span $T(P)$, so that for any choice of x^*, y^* the point (x^*, y^*, z^*) lies in $T(P)$. Also, $x^* = x$, $y^* = y$ gives $z^* = z$, so that $T(P)$ passes through P, as it should.

SECTION 9.6. Surface Integrals, page 496

Purpose. We define and discuss surface integrals with and without taking into account surface orientations.

Main Content

Surface integrals (3) $\equiv$ (4) $\equiv$ (6)

Change of orientation (Theorem 1)

Integrals (7) without regard to orientation; also (11)

Comments on Content

The right side of (3) shows that we need only $\mathbf{N}$ but not the corresponding unit vector $\mathbf{n}$.

An orientation results automatically from the choice of a surface representation, which determines $\mathbf{r}_u$ and $\mathbf{r}_v$ and thus $\mathbf{N}$.

The existence of nonorientable surfaces is interesting, but is not needed in our further work.

SOLUTIONS TO PROBLEM SET 9.6, page 503

2. $\mathbf{r} = [u,\ \ v,\ \ 1 - u - v]$, $0 \leqq u \leqq 1 - v$, $0 \leqq v \leqq 1$, $\mathbf{F}(r) = [u^2,\ \ e^v,\ \ 1]$, $\mathbf{N} = [1,\ \ 1,\ \ 1]$ (which is obvious without calculation), $\mathbf{F}(r) \bullet \mathbf{N} = u^2 + e^v + 1$. *Answer:* $e - 17/12 \approx 1.30$

4. $\mathbf{F}(\mathbf{r}) = [\sinh(\cos v \sin v),\ \ 0,\ \ \cos^4 v]$, $\mathbf{N} = [0,\ \ -\cos v,\ \ -\sin v]$, $\mathbf{F}(\mathbf{r}) \bullet \mathbf{N} = -\cos^4 v \sin v$. *Answer:* $-16/5$

6. $\mathbf{r} = [\cos u,\ \ \frac{1}{2}\sin u,\ \ v]$, $\mathbf{F}(\mathbf{r}) = [\frac{1}{8}\sin^3 u,\ \ \cos^3 u,\ \ v^3]$, $\mathbf{N} = [\frac{1}{2}\cos u,\ \ \sin u,\ \ 0]$, $\mathbf{F}(\mathbf{r}) \bullet \mathbf{N} = \cos^3 u \sin u + \frac{1}{16}\cos u \sin^3 u$, $0 \leqq u \leqq \pi/2$, $0 \leqq v \leqq h$. *Answer:* $17h/64$

8. $\mathbf{N} = [\cosh u,\ \ -\sinh u,\ \ 0]$, $\mathbf{F}(\mathbf{r}) \bullet \mathbf{N} = 2\cosh^2 u \sinh u$. *Answer:* $4\cosh^3 2 - 4$

10. $\mathbf{F}(\mathbf{r}) = [u^2 \cos^2 v,\ \ u^2 \sin^2 v,\ \ 9v^2]$, $\mathbf{N} = [3\sin v,\ \ -3\cos v,\ \ u]$, $\mathbf{F}(\mathbf{r}) \bullet \mathbf{N} = 9uv^2 + 3u^2(\cos^2 v \sin v - \cos v \sin^2 v)$. The integral of $9uv^2$ is $12\pi^3$. The integral of the other term is zero. *Answer:* $12\pi^3$

12. $x = u, y = v, \quad G = \cos u + \sin v, \quad |\mathbf{N}| = \sqrt{3}$, and

$$\int_0^1 \int_0^{1-u} (\cos u + \sin v)\, dv\, du = 2 - \cos 1 - \sin 1.$$

Answer: $(2 - \cos 1 - \sin 1)\sqrt{3}$.

14. $\mathbf{N} = [5 \cos u, \quad 5 \sin u, \quad 0]$, $|\mathbf{N}| = 5$, $G(\mathbf{r})|\mathbf{N}| = 5 \cdot 625(\cos^4 u + \sin^4 u)$. Integration over u gives $18750\pi/8$. Integration over v then gives the answer $937.5\pi \approx 2945$.

16. $\mathbf{N} = [-2u \cos v, \quad -2u \sin v, \quad u]$, $|\mathbf{N}| = u\sqrt{5}$, $G(\mathbf{r})|\mathbf{N}| = \sqrt{5}(u^5 - 4u^3)$. *Answer:* $-5\sqrt{5}\pi/3$

20. $I_{x=y} = \iint_S \left[\frac{1}{2}(x - y)^2 + z^2\right] \sigma\, dA$

22. $h\pi(1 + h^2/6)$

24. Proof for a lamina S of density σ. Choose coordinates so that A is the z-axis and B is the line $x = k$ in the xz-plane. Then

$$\begin{aligned} I_B &= \iint_S [(x - k)^2 + y^2]\sigma\, dA = \iint_S [x^2 - 2kx + k^2 + y^2]\sigma\, dA \\ &= \iint_S (x^2 + y^2)\sigma\, dA - 2k \iint_S x\sigma\, dA + k^2 \iint_S \sigma\, dA \\ &= I_A - 2k \cdot 0 + k^2 M, \end{aligned}$$

the second integral being zero because it is the first moment of the mass about an axis through the center of gravity.

For a mass distributed in a region in space the idea of proof is the same.

26. TEAM PROJECT. (a) Use $d\mathbf{r} = \mathbf{r}_u\, du + \mathbf{r}_v\, dv$. This gives (13) and (14) because

$$d\mathbf{r} \cdot d\mathbf{r} = \mathbf{r}_u \cdot \mathbf{r}_u\, du^2 + 2\mathbf{r}_u \cdot \mathbf{r}_v\, du\, dv + \mathbf{r}_v \cdot \mathbf{r}_v\, dv^2.$$

(b) E, F, G appear if you express everything in terms of dot products. In the numerator,

$$\mathbf{a} \cdot \mathbf{b} = (\mathbf{r}_u g' + \mathbf{r}_v h') \cdot (\mathbf{r}_u p' + \mathbf{r}_v q') = Eg'p' + F(g'q' + h'p') + Gh'q'$$

and similarly in the denominator.

(c) This follows by Lagrange's identity (Problem Set 8.3),

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v|^2 = (\mathbf{r}_u \times \mathbf{r}_v) \cdot (\mathbf{r}_u \times \mathbf{r}_v) &= (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)^2 \\ &= EG - F^2 \end{aligned}$$

(d) $\mathbf{r} = [u \cos v, \quad u \sin v]$, $\mathbf{r}_u = [\cos v, \quad \sin v]$, $\mathbf{r}_u \cdot \mathbf{r}_u = \cos^2 v + \sin^2 v = 1$, etc.

(e) By straightforward calculation $E = (a + b \cos v)^2$, $F = 0$ (the coordinate curves on the torus are orthogonal!), and $G = b^2$. Hence, as expected,

$$\sqrt{EG - F^2} = b(a + b \cos v).$$

SECTION 9.7. Triple Integrals. Divergence Theorem of Gauss, page 505

Purpose, Content

Proof and application of the first "big" integral theorem in this chapter, Gauss's theorem, preceded by a short discussion of triple integrals (probably known to most students from calculus).

Comment on Proof

The proof is simple:

1. Cut (2) into three components. Take the third, (6).

2. On the left, integrate $\iiint \frac{\partial F_3}{\partial z}\, dz\, dx\, dy$ over z to get

$$\iint [F_3(\text{upper surface}) - F_3(\text{lower surface})]\, dx\, dy \tag{9}$$

integrated over the projection R of the region in the xy-plane (Fig. 231).

3. Show that the right side of (6) equals (9). Since the third component of $\mathbf{n}$ is $\cos\gamma$, the right side is

$$\iint F_3 \cos\gamma\, dA = \iint F_3\, dx\, dy$$

$$= \iint F_3(\text{upper})\, dx\, dy - \iint F_3(\text{lower})\, dx\, dy,$$

where minus comes from $\cos\gamma < 0$ in Fig. 231, lower surface. This is the proof. Everything else is (necessary) accessory.

SOLUTIONS TO PROBLEM SET 9.7, page 509

2. Integration over x, y, z gives successively $-e^{-1-z} + e^{-y-z}$, $-2e^{-1-z} + e^{-z}$, $2e^{-3} - e^{-2} - 2e^{-1} + 1$.

4. In polar coordinates, $\sigma = r^4/3$. The integral is

$$\int_{-3}^{3}\int_0^{2\pi}\int_0^3 \frac{1}{3} r^5\, dr\, d\theta\, dz = 486\pi \approx 1527.$$

6. We may integrate in the order $0 \leqq z \leqq x$, $0 \leqq y \leqq 1 - x^2$, $0 \leqq x \leqq 1$. This gives $2x^2$, then $2x^2(1 - x^2)$, and finally the *answer* 4/15.

8. $abc(b^2 + c^2)/12$ 10. $\pi h^5/10$ 12. $8\pi a^5/15$

14. $\operatorname{div}\mathbf{F} = e^x + e^y + e^z$. Integration over x gives $2\sinh 1 + 2e^y + 2e^z$. Integration over y then gives $4\sinh 1 + 4\sinh 1 + 4e^z$. Integration over z gives the *answer* $24\sinh 1$.

16. $\operatorname{div}\mathbf{F} = -\sin z$. Integration over z gives $\cos 2 - 1$. Multiplication by the cross-sectional area 9π gives the *answer* $9\pi(\cos 2 - 1) \approx -40.04$.

18. $\operatorname{div}\mathbf{F} = 4x + y + \pi\sin\pi z$. Integration over z from 0 to $1 - x - y$ gives $1 + (1 - x - y)(4x + y) - \cos[\pi(1 - x - y)]$. Then integration over y from 0 to $1 - x$ gives

$$\frac{7}{6} + \frac{1}{2}x - \frac{7}{2}x^2 + \frac{11}{6}x^3 - \frac{1}{\pi}\sin[\pi(1 - x)].$$

Integration over x from 0 to 1 now gives the *answer* $17/24 - 2/\pi^2$.

SECTION 9.8. Further Applications of the Divergence Theorem, page 510

Purpose. To represent the divergence free of coordinates (Example 1), to show that it measures the source intensity (Example 2), to use Gauss's theorem for deriving the

heat equation governing heat flow in a region, and to obtain basic properties of harmonic functions.

Main Content, Important Concepts

Divergence as the limit of a surface integral; see (3)

Total flow (4) out of a region

Heat equation (7) (to be discussed further in Chap. 11)

Properties of harmonic functions (Theorems 1–3)

Green's formulas (10), (11)

Short Courses. This section can be omitted.

Comments on (3)

Equation (3) is sometimes used as a *definition* of the divergence, giving independence of the choice of coordinates immediately. Also, Gauss's theorem follows more readily, but since its proof is simple (see Sec. 9.7. in this Manual), that savings is marginal. Also, it seems that to the student our Example 2 in Sec. 8.10 motivates the divergence at least as well (and without integrals) as (3) does for a beginner.

SOLUTIONS TO PROBLEM SET 9.8, page 514

2. $\mathbf{r} = [2\cos\theta,\ 2\sin\theta] = \mathbf{N}$, $\mathbf{n} = [\cos\theta,\ \sin\theta]$, $f = 4\cos^2\theta - 4\sin^2\theta$, $\partial f/\partial n = \nabla f \cdot \mathbf{n} = 4\cos^2\theta - 4\sin^2\theta = 4\cos 2\theta$ gives the integral zero. The integrals over the disks ($z = 0$ and $z = 1$) are zero, too, since ∇f has no component in z-direction (the normal direction of those disks).

4. $\operatorname{div}\mathbf{F} = 10$. *Answer:* 10 times the volume $\pi r^2 h/3$ of a cone of base radius r and height h, thus 90π.

6. $\operatorname{div}\mathbf{F} = 10 + 3z^2$. Hence

$$\int_{-1}^{1}\int_{x/2}^{x}\int_{0}^{y}(10 + 3z^2)\,dz\,dy\,dx = \int_{-1}^{1}\int_{x/2}^{x}(10y + y^3)\,dy\,dx =$$

$$\int_{-1}^{1}\left(\frac{15}{4}x^2 + \frac{15}{64}x^4\right)dx = \frac{83}{32}.$$

8. $\operatorname{div}\mathbf{F} = x + y$. (a) In polar coordinates (cylindrical coordinates)

$$\int_0^4\int_0^{\pi}\int_0^{\sqrt{z}}(r\cos\theta + r\sin\theta)r\,dr\,d\theta\,dz = \int_0^4 \frac{2}{3}z^{3/2}\,dz = \frac{128}{15}.$$

(b) In Cartesian coordinates,

$$\int_0^4\int_{-\sqrt{z}}^{\sqrt{z}}\int_0^{\sqrt{z-x^2}}(x + y)\,dy\,dx\,dz = \int_0^4\int_{-\sqrt{z}}^{\sqrt{z}}\left(x\sqrt{z - x^2} + \frac{1}{2}(z - x^2)\right)dx\,dz$$

$$= \int_0^4\left(0 + \frac{2}{3}z^{3/2}\right)dz = \frac{128}{15}.$$

10. $r = a$, $\cos\phi = 1$; $V = \frac{1}{3}a(4\pi a^2)$

12. TEAM PROJECT. (a) Put $f = g$ in (10).

(b) Use (a).

(c) Use (11).

(d) $h = f - g$ is harmonic and $\partial h/\partial n = 0$ on S. Thus $h = const$ in T by (b).
(e) Use div grad $f = \nabla^2 f$.

SECTION 9.9. Stokes's Theorem, page 515

Purpose. To prove, explain, and apply Stokes's theorem, relating line and surface integrals.

Main Content

Formula (2) ≡ (3)

Further interpretation of the curl (see also Sec. 8.11)

Path independence of line integrals (leftover from Sec. 9.2)

Comment on Orientation

Since the choice of right-handed or left-handed coordinates is essential to the curl (Sec. 8.11), surface orientation becomes essential here (Fig. 232).

Comment on Proof

The proof is simple:

1. Cut (3) into components. Take the first, (4).
2. Cast the left side of (4) by using N_1, N_3 into the form (8).
3. Transform the right side of (4) by Green's theorem into a double integral and show equality with the integral obtained on the left.

SOLUTIONS TO PROBLEM SET 9.9, page 520

2. curl $\mathbf{F} = (-2x - 2y)\mathbf{k}$,

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (-2x - 2y)\,dx\,dy = \int_0^2 -4y\sqrt{4 - y^2}\,dy = -\frac{32}{3}.$$

Line integral: Over the x-axis from -2 to 2 we obtain

$$\int_{-2}^{2} -x^2\,dx = -\frac{16}{3}$$

and over the semicircle $\mathbf{r} = [2\cos t,\ \ 2\sin t,\ \ 0]$, thus $\mathbf{r}' = [-2\sin t,\ \ 2\cos t,\ \ 0]$, we obtain

$$\int_0^{\pi} \mathbf{F}(\mathbf{r})\cdot\mathbf{r}'\,dt = \int_0^{\pi} [(4\cos^2 t)(-2\sin t) - (4\sin^2 t)(2\cos t)]\,dt$$

$$= \frac{8}{3}\cos^3 t\,\Big|_0^{\pi} - \frac{8}{3}\sin^3 t\,\Big|_0^{\pi} = -\frac{16}{3}.$$

The verifications in Probs. 1–6 are supposed to familiarize the student more thoroughly with the nature and significance of Stokes's theorem.

4. curl $\mathbf{F} = -\cos 2z\,\mathbf{j}$, S: $\mathbf{r} = [\cos u,\ \ \sin u,\ \ v]$, $\mathbf{N} = [\cos u,\ \ \sin u,\ \ 0]$; thus

$$\int_0^{\pi/4} \int_0^{\pi} -\cos 2v \sin u\,du\,dv = -1.$$

Line integral: The two circular arcs contribute nothing, as can be seen from **F**, whose first two components are zero. From $(-1, 0, 0)$ straight up to $(-1, 0, \pi/4)$ we get

$$\int_0^{\pi/4} -1 \cdot \cos 2z\, dz = -\tfrac{1}{2}.$$

Similarly, $-\frac{1}{2}$ is obtained for the straight-line segment from $(1, 0, \pi/4)$ to $(1, 0, 0)$.

6. $\operatorname{curl} \mathbf{F} = [-2z,\ -2x,\ -2y]$, S: $\mathbf{r} = [\cos v \cos u,\ \cos v \sin u,\ 1 + \sin v]$, $0 \leqq u \leqq \pi$, $-\pi/2 \leqq v \leqq 0$, $\mathbf{N} = [\cos^2 v, \cos u,\ \cos^2 v \sin u,\ \cos v \sin v]$

$$\int_{-\pi/2}^{0}\int_0^{\pi} (-\cos^3 v \sin 2u - 2\cos u \cos^2 v\,(1 + \sin v) - \cos v \sin u \sin 2v)\, du\, dv$$

$$= \int_{-\pi/2}^{0} -4\cos^2 v \sin v\, dv = \tfrac{4}{3}.$$

Verification: The line integral over the horizontal semicircle equals $\pm 4/3$, as in Prob. 5. Over the vertical semicircle it is zero because $\mathbf{r} = [t,\ 0,\ \sqrt{1 - t^2}]$, $\mathbf{r}' = [1,\ 0,\ -t/\sqrt{1 - t^2}]$; on the semicircle, $\mathbf{F} = [0,\ 1 - t^2,\ t^2]$, so that the integrand is $t^3/\sqrt{1 - t^2}$, and the integral is zero, as claimed.

8. $\operatorname{curl} \mathbf{F} = (1 - 4y)\mathbf{k}$, $\mathbf{N} = \mathbf{k}$. Hence, writing $X = \sqrt{a^2 - x^2}$, we have

$$\int_{-a}^{a}\int_{-X}^{X} (1 - 4y)\, dy\, dx = \int_{-a}^{a} 2X\, dx = 2a^2 \arcsin 1 = \pi a^2.$$

10. $\operatorname{curl} \mathbf{F} = [1, 0, -1]$, $\mathbf{N} = [-1, 0, 1]$, $(\operatorname{curl} \mathbf{F}) \bullet \mathbf{n} = -\sqrt{2}$. Multiplication by the area $9\pi\sqrt{2}$ gives the *answer* -18π.

12. $\operatorname{curl} \mathbf{F} = \mathbf{0}$. *Answer:* 0

14. $\mathbf{r} = [\cos\theta,\ \sin\theta,\ 0]$. Hence

$$\mathbf{F} \bullet \mathbf{r}' = [-\sin\theta,\ \cos\theta,\ 0] \bullet [-\sin\theta,\ \cos\theta,\ 0] = 1. \text{ Answer: } 2\pi.$$

SOLUTIONS TO CHAPTER 9 REVIEW, page 521

16. $-325/3$ by exactness from $f = (x^3 - 2y^3)/3$, or by integration from $\mathbf{r} = [4 - 7t,\ 2 + 3t]$, $\mathbf{r}' = [-7,\ 3]$,

$$\int_0^1 ((4 - 7t)^2 \cdot (-7) - 2(2 + 3t)^2 \cdot 3)\, dt = -\frac{325}{3}.$$

18. Not exact, C: $\mathbf{r} = [x,\ 2x^2,\ x]$, $\mathbf{r}' = [1,\ 4x,\ 1]$. $\mathbf{F}(\mathbf{r}) = [2x^3,\ x,\ 0]$. Hence

$$\int_1^2 (2x^3 + 4x^2)\, dx = \frac{101}{6}.$$

20. Not exact. By Green's theorem in the plane, using polar coordinates, we obtain

$$\int_0^{2\pi}\int_0^2 3r^2 r\, dr\, d\theta = 24\pi. \qquad \text{Answer: } \pm 24\pi$$

22. $\operatorname{curl} \mathbf{F} = [-1,\ -1,\ 5]$, $\mathbf{N} = \pm[-1,\ 0,\ 1]$, $(\operatorname{curl} \mathbf{F}) \bullet \mathbf{n} = \pm 6/\sqrt{2}$. The area of the ellipse is $\pi \cdot 2 \cdot 2\sqrt{2}$. *Answer:* $\pm 24\pi \approx 75.4$

24. Exact, $e - e^{-1} = 2 \sinh 1$ from $f = e^{xz} + \cosh 2y$

26. By Green's theorem in the plane,

$$\int_0^2 \int_1^2 \left(2ye^x - \frac{x}{y}\right) dy\, dx = \int_0^2 (4e^x - x \ln 2 - e^x)\, dx$$

$$= 3e^2 - 2 \ln 2 - 3.$$

Answer: $\pm(3e^2 - 2 \ln 2 - 3)$.

28. Not exact, $\frac{1}{3} \sinh 3 + \frac{29}{4} \approx 10.59$

30. Not exact, $\mathbf{r}' = [-\sin t, \; \cos t, \; 3]$, $\mathbf{F}(\mathbf{r}) = [\cos^2 t, \; \sin^2 t, \; \sin^2 t \cos t]$, $\mathbf{F}(\mathbf{r}) \cdot \mathbf{r}' = -\cos^2 t \sin t + 4 \sin^2 t \cos t$. *Answer:* 1

32. $M = \int_{-1}^{1} \int_{x^2}^{1} f\, dy\, dx = \frac{4}{21}, \quad \bar{x} = 0, \quad \bar{y} = \frac{21}{4} \int_{-1}^{1} \int_{x^2}^{1} x^2 y^2\, dy\, dx = \frac{7}{9}$

34. $M = \pi a^2/2, \quad \bar{x} = 0$ (symmetry), $\quad \bar{y} = \frac{1}{M} \int_0^{\pi} \int_0^{a} r \sin \theta\, r\, dr\, d\theta = \frac{4a}{3\pi}$

36. $M = \int_0^{\pi/2} \int_0^{a} r^3\, dr\, d\theta = \pi a^4/8, \quad \bar{x} = (1/M) a^5/5 = 8a/5\pi, \quad \bar{y} = 8a/5\pi$

38. $\mathbf{r} = [v, \; 2 \cos u, \; 2 \sin u]$, $\mathbf{F}(\mathbf{r}) = [\sin v, \; 2 \sin u, \; 2 \cos u]$, $\mathbf{N} = [0, \; 2 \cos u, \; 2 \sin u]$, $\mathbf{F}(\mathbf{r}) \cdot \mathbf{N} = 8 \cos u \sin u$. *Answer:* 4

40. $\operatorname{div} \mathbf{F} = 3(x^2 + y^2 + z^2)$. Hence in Cartesian coordinates,

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} 3(x^2 + y^2 + z^2)\, dz\, dy\, dx = \frac{384\pi}{5}.$$

In spherical coordinates, with $dx\, dy\, dz = r^2 \sin \phi\, dr\, d\phi\, d\theta$,

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{3} 3r^2 r^2\, dr \sin \phi\, d\phi\, d\theta = \frac{384\pi}{5}.$$

42. $\mathbf{N} = [2, \; -1, \; 0]$, $\mathbf{F}(\mathbf{r}) = [e^{2u}, \; 0, \; ve^u]$, $\mathbf{F}(\mathbf{r}) \cdot \mathbf{N} = 2e^{2u}$. *Answer:* 6 sinh 2

44. $\operatorname{div} \mathbf{F} = 2z, \; \int_0^2 2z \int_0^2 \int_0^2 r\, dr\, d\theta\, dz = 16\pi$

PART C. FOURIER ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS

CHAPTER 10 Fourier Series, Integrals, and Transforms

Change

The presentation was streamlined by moving half-range expansions into the section on even and odd functions.

SECTION 10.1. Periodic Functions. Trigonometric Series, page 527

Purpose. To show what a Fourier series will look like; in Problem Set 10.1, to give a first impression of what kind of functions will occur in this chapter.

Basic Concepts

Periodic function

Trigonometric system

Trigonometric series

Comment on Footnote 1

Fourier series were used in special problems much earlier by Daniel Bernoulli (1700—1782) in 1748 (vibrating string, Sec. 11.3) and Euler (Sec. 2.6) in 1754 (Euler formulas, Sec. 10.2). Fourier's book of 1822 became the source of many mathematical methods in classical mathematical physics. Furthermore, the surprising fact that Fourier series, whose terms are *continuous* functions, may represent *discontinuous* functions led to a reflection on, and generalization of, the concept of a function in general. Hence the book is a landmark in both pure and applied mathematics. [That surprising fact also led to a controversy between Euler and D. Bernoulli over the question of whether the two types of solution of the vibrating string problem (Secs. 11.3 and 11.4) are identical; for details, see E. T. Bell, *The Development of Mathematics,* New York: McGraw-Hill, 1940, p. 482.] A mathematical theory of Fourier series was started by Peter Gustav Lejeune Dirichlet (1805—1859) of Berlin in 1829. The concept of the Riemann integral also resulted from work on Fourier series. Later on, these series became the model case in the theory of orthogonal functions (Sec. 4.7). An English translation of Fourier's book was published by Dover Publications in 1955.

SOLUTIONS TO PROBLEM SET 10.1, page 528

2. $2\pi/n$, $2\pi/n$, k, k, k/n, k/n

4. True when $n = 1$. Induction hypothesis $f(x + np) = f(x)$. Now set $x + np = z$. Then $f(x + (n + 1)p) = f(z + p) = f(z) = f(x)$.

6. $f(x + p) = f(x)$ implies $f(ax + p) = f(a[x + (p/a)]) = f(ax)$ or $g[x + (p/a)] = g(x)$, where $g(x) = f(ax)$. Thus $g(x)$ has period p/a. This proves the first statement, and the other statement follows by setting $a = 1/b$.

8. A common **source of errors** here and throughout this chapter results from the fact that the student often does not pay attention to the interval on which the function is given, notably whether it is $-\pi \leqq x \leqq \pi$ or $0 \leqq x \leqq 2\pi$.

Note that the first of these is preferable because it shows more immediately whether a function is odd or even (or neither).

20. CAS PROJECT. This "experimental approach" to trigonometric and Fourier series should help the student obtain a feeling for the kind of series to expect in practice, and for the kind and quality of convergence, depending on continuity properties of the sum of the series.

Convergence is best for the first of the three series because its sum x^2 is continuous—note that the coefficients are proportional to $1/n^2$, whereas for the other two series they are only proportional to $1/n$. The second series has the square wave in Prob. 15 as its sum. The third series has the sum $f(x) = x$. Hopefully it will puzzle the student by its poor convergence behavior (and the Gibbs phenomenon) near the discontinuity points at $x = \pm\pi$, where it converges to the mean of the left-hand and right-hand limits, which is typical (see Sec. 10.2).

SECTION 10.2. Fourier Series, page 529

Purpose. To derive the Euler formulas (6) for the coefficients of a Fourier series (7) of a given function of period 2π, using the key property of the orthogonality of the trigonometric system.

Main Content, Important Concepts

Euler formulas (6) for Fourier coefficients (period 2π)

Orthogonality of the trigonometric system

Convergence and sum of a Fourier series (Theorem 1)

Comment on Notation

If we write $a_0/2$ instead of a_0 in (1), we must do the same in (6a) and see that (6a) then becomes (6b) with $n = 0$. This is merely a small notational convenience (but may be a source of confusion to poorer students).

Comment on Fourier Series

Whereas their theory is quite involved, practical applications are simple, once the student has become used to evaluating integrals in (6) that depend on n.

Figure 238 should help students understand why and how a series of continuous terms can have a discontinuous sum.

SOLUTIONS TO PROBLEM SET 10.2, page 536

2. $$\frac{1}{4} + \frac{1}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - + \cdots\right) + \frac{1}{\pi}\left(\sin x + \frac{2}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{2}{6}\sin 6x + \cdots\right)$$

4. $$-\frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - + \cdots\right) + \frac{2}{\pi}\left(\sin 2x + \frac{1}{3}\sin 6x + \cdots\right)$$

6. $$\pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

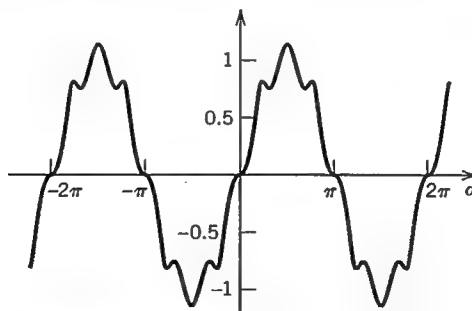
8. $\frac{4\pi^2}{3} + 4\left(\cos x + \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x + \cdots\right)$
$$- 4\pi\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

10. $\frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$
$$+ 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + - \cdots\right)$$

12. $-\frac{1}{4} - \frac{1}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - + \cdots\right)$
$$-\frac{1}{\pi}\left(\sin x + \frac{2}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{2}{6}\sin 6x + \cdots\right)$$

14. $\frac{4}{\pi}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - + \cdots\right)$

16. $\frac{\pi^2}{6} - \frac{4}{\pi}\cos x - \frac{2}{2^2}\cos 2x + \frac{4}{3^3\pi}\cos 3x + \frac{2}{4^2}\cos 4x - \frac{4}{5^3\pi}\cos 5x + \cdots$

18. The student should be encouraged to choose any other integrals of products of cosines and sines. The point is to realize the importance of the interval in connection with orthogonality. The integral suggested in the problem has the value $\sin a - \frac{1}{7}\sin 7a$. The figure suggests orthogonality for $a = \pi$, as expected.

Section 10.2. Integral in Problem 18

SECTION 10.3. Functions of Any Period $p = 2L$, page 537

Purpose. Transition from period 2π to period $2L$ (a notation practical later), simply by a linear transformation on the x-axis, giving the Fourier series (1) with coefficients (2).

SOLUTIONS TO PROBLEM SET 10.3, page 540

2. -1 times the answer to Prob. 1

4. $1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{9}\cos\frac{3\pi x}{2} + \frac{1}{25}\cos\frac{5\pi x}{2} + \cdots\right)$

6. $\frac{2}{3} + \frac{4}{\pi^2}\left(\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - + \cdots\right)$

8. $\frac{1}{4} + \frac{2}{\pi^2}\left(\cos 2\pi x + \frac{1}{9}\cos 6\pi x + \frac{1}{25}\cos 10\pi x + \cdots\right)$

10. $-\frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right)$

$$+ \frac{2}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \cdots\right)$$

12. $\left(1 - \frac{6}{\pi^2}\right)\sin \pi x - \left(\frac{1}{2} - \frac{6}{2^3\pi^2}\right)\sin 2\pi x + \left(\frac{1}{3} - \frac{6}{3^3\pi^2}\right)\sin 3\pi x - + \cdots$

14. $b_n = 0,\ a_0 = \frac{V_0}{\pi}$,

$$a_n = 100\int_{-1/200}^{1/200} V_0 \cos 100\pi t \cos 100n\pi t\, dt$$

$$= 50V_0\int_{-1/200}^{1/200} \cos 100(n+1)\pi t\, dt + 50V_0\int_{-1/200}^{1/200} \cos 100(n-1)\pi t\, dt,$$

$$\frac{V_0}{\pi} + \frac{V_0}{2}\cos 100\pi t$$

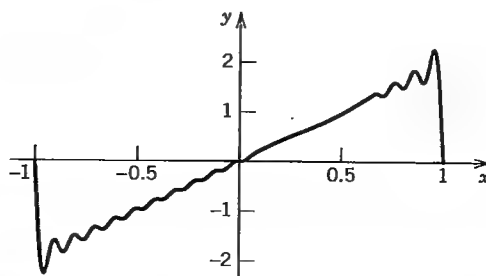
$$+ \frac{2V_0}{\pi}\left(\frac{1}{1\cdot 3}\cos 200\pi t - \frac{1}{3\cdot 5}\cos 400\pi t + \frac{1}{5\cdot 7}\cos 600\pi t - + \cdots\right)$$

16. In Prob. 7, Sec. 10.2, write t for x and

$$\tilde{f}(t) = t^2 = \frac{\pi^2}{3} - 4\left(\cos t - \frac{1}{4}\cos 2t + - \cdots\right).$$

Now set $t = \pi x$ to get $\tilde{f}(t) = \pi^2 x^2$, which shows that the series should be multiplied by $3/\pi^2$ to get that of

$$f(x) = 3x^2 = 1 - \frac{12}{\pi^2}\left(\cos \pi x - \frac{1}{4}\cos 2\pi x + - \cdots\right).$$

20. CAS PROJECT. The figure shows $s_{20}(x)$.

Section 10.3. Gibbs phenomenon in CAS Project 20

SECTION 10.4. Even and Odd Functions. Half-Range Expansions, page 541

Purpose. 1. To show that a Fourier series of an even function (an odd function) has only cosine terms (only sine terms), so that unnecessary work (and sources of errors!) is avoided.

2. To represent a function $f(x)$ by a Fourier cosine series or by a Fourier sine series (of period $2L$) if $f(x)$ is given on an interval $0 \leqq x \leqq L$ only, which is half the interval of periodicity—hence the name "half-range."

Comment

Such half-range expansions occur in vibrational problems, heat problems, etc., as will be shown in Chap. 11.

SOLUTIONS TO PROBLEM SET 10.4, page 546

2. Even: $|x|$, e^{x^2}, $\sin^2 x$, $x \sin x$, $e^{-|x|}$. Odd: $x \cos x$.

4. Neither even nor odd. The problem shows that the student must always pay careful attention to the interval on which the function is given because for different functions, different intervals are practical.

6. Even **8.** Neither even nor odd

10. PROJECT. (a) Sums and products of even functions are even. Sums of odd functions are odd. Products of odd functions are odd (even) if the number of their factors is odd (even). Products of an even times and odd function are odd. This is important in connection with the integrands in the Euler formulas for the Fourier coefficients. Absolute values of odd functions are even. $f(x) + f(-x)$ is even, $f(x) - f(-x)$ is odd.

(b) $e^{kx} = \cosh kx + \sinh kx, \quad \dfrac{1}{1-x} = \dfrac{1}{1-x^2} + \dfrac{x}{1-x^2}$; furthermore,

$$\sin(x + k) = \sin k \cos x + \cos k \sin x,$$

$$\cosh(x + k) = \cosh k \cosh x + \sinh k \sinh x.$$

(c) $f(-x) = -f(x)$ and $f(-x) = f(x)$ together imply $f = 0$.

(d) $\cos^3 x$ is even, $\sin^3 x$ is odd. The Fourier series are the familiar identities

$$\cos^3 x = \tfrac{3}{4}\cos x + \tfrac{1}{4}\cos 3x \qquad \text{and} \qquad \sin^3 x = \tfrac{3}{4}\sin x - \tfrac{1}{4}\sin 3x.$$

12. $\pi - \dfrac{8}{\pi}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \cdots\right)$

14. $-\dfrac{4}{\pi}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \cdots\right) + 2\left(\sin x + \dfrac{1}{3}\sin 3x + \cdots\right)$

16. $36\left[\dfrac{1}{1^3}\sin x - \dfrac{1}{2^3}\sin 2x + \dfrac{1}{3^3}\sin 3x - + \cdots\right]$

18. Set $x = \pi$ to get

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right).$$

The result was first obtained by Euler.

20. The even periodic extension has the Fourier series $f(x) = 1$. For the odd half-range expansion we obtain

$$\frac{4}{\pi}\left(\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \cdots\right).$$

22. The cosine series is

$$\frac{L^2}{3} - \frac{4L^2}{\pi^2}\left(\cos\frac{\pi x}{L} - \frac{1}{4}\cos\frac{2\pi x}{L} + \frac{1}{9}\cos\frac{3\pi x}{L} - \frac{1}{16}\cos\frac{4\pi x}{L} + - \cdots\right).$$

Its Fourier coefficients are proportional to $1/n^2$, reflecting the fact that its sum is continuous. The sine series is

$$\frac{2L^2}{\pi}\left[\left(1 - \frac{4}{\pi^2}\right)\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \left(\frac{1}{3} - \frac{4}{3^3\pi^2}\right)\sin\frac{3\pi x}{L} - \frac{1}{4}\sin\frac{4\pi x}{L} + \cdots\right].$$

Its coefficients are only proportional to $1/n$, reflecting the fact that its sum is discontinuous.

24. The cosine series is

$$\frac{L^3}{4} + \frac{6L^3}{\pi^2}\left[\left(\frac{4}{\pi^2} - 1\right)\cos\frac{\pi x}{L} + \frac{1}{2^2}\cos\frac{2\pi x}{L} + \left(\frac{4}{3^4\pi^2} - \frac{1}{3^2}\right)\cos\frac{3\pi x}{L} + \cdots\right].$$

Its sum is continuous. Its coefficients go to zero faster than those of the sine series

$$\frac{2L^3}{\pi^3}\left[\left(\frac{\pi^2}{1} - \frac{6}{1^3}\right)\sin\frac{\pi x}{L} - \left(\frac{\pi^2}{2} - \frac{6}{2^3}\right)\sin\frac{2\pi x}{L} + \left(\frac{\pi^2}{3} - \frac{6}{3^3}\right)\sin\frac{3\pi x}{L} - + \cdots\right]$$

whose sum is discontinuous.

SECTION 10.5. Complex Fourier Series. *Optional,* page 547

Purpose. To show that the formula for $e^{i\theta}$ or direct derivation leads to the complex Fourier series in which complex exponential functions (instead of cosine and sine) appear. This is interesting, but will not be needed in our further work, so that we can leave it optional.

Short Courses. Sections 10.5–10.11 can be omitted.

SOLUTIONS TO PROBLEM SET 10.5, page 549

2. $-\frac{2i}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{2n+1}e^{(2n+1)ix}$

4. $\frac{1}{2} - \frac{i}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{2n+1}e^{(2n+1)ix}$

6. $\frac{\pi^2}{3} + 2\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{(-1)^n}{n^2}e^{inx}$

8. See Prob. 6, Sec. 10.2.

10. PROJECT. When $n = m$, the integrand of the integral on the right is $e^0 = 1$, so that the integral equals 2π. This gives

$$\text{(A)}\qquad \int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = 2\pi c_m,$$

provided the other integrals are zero, which is true by (5),

$$\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \frac{1}{i(n-m)}\left(e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right)$$

$$= \frac{1}{i(n-m)}\,2i\sin(n-m)\pi = 0.$$

Now writing n for m in (A) gives the coefficient formula in (8).

SECTION 10.6. Forced Oscillations, page 550

Purpose. To show that mechanical or electrical systems with periodic but nonsinusoidal input may respond predominantly to one of the infinitely many terms in the Fourier series of the input, giving an unexpected output; see Fig. 252, where the output frequency is essentially five times that of the input.

Short Courses. Sections 10.5–10.11 can be omitted.

SOLUTIONS TO PROBLEM SET 10.6, page 552

2. $r'(t)$ is given by the sine series in Example 1, Sec. 10.2, with $k = -1$. The new C_n is n times the old, so that C_5 is so large that the output is practically a cosine vibration having five times the input frequency. Replacement of the right side by its integral (with $r(0) = 0$) also produces an increase of C_5.

4. $y = C_1 \cos \omega t + C_2 \sin \omega t + \dfrac{1}{\omega^2 - \alpha^2}\cos \alpha t + \dfrac{1}{\omega^2 - \beta^2}\cos \beta t$

6. $y = C_1 \cos \omega t + C_2 \sin \omega t + B_1 \sin t + B_3 \sin 3t + B_5 \sin 5t$, where

ω	0.5	0.9	1.1	2.0	2.9	3.1	4.0	4.9	5.1	6.0	8.0
$B_1 = 1/(\omega^2 - 1)$	−1.33	−5.3	4.8	0.33	0.13	0.12	0.07	0.04	0.04	0.03	0.02
$B_3 = 1/9(\omega^2 - 9)$	−0.013	−0.014	−0.014	−0.02	−0.19	0.18	0.02	0.01	0.01	0.004	0.002
$B_5 = 1/25(\omega^2 - 25)$	−0.002	−0.002	−0.002	−0.002	−0.002	−0.003	−0.004	−0.04	0.04	0.004	0.001

$\omega/2\pi$ is the frequency of the free vibration. If ω comes close to 1, 3, or 5 in one of the terms of the input $r(t)$, then the output corresponding to that term becomes comparatively large in amplitude, B_1 if ω is near 1, next B_3 if ω is near 3, and finally B_5 if ω is near 5. The effect would even be stronger had we chosen all coefficients on the right equal to 1, instead of $1/n^2$ as in the partial sum of a Fourier series.

8. $y = C_1 \cos \omega t + C_2 \sin \omega t + \dfrac{1}{2\omega^2} + \dfrac{1}{1 \cdot 3(\omega^2 - 4)}\cos 2t$

$$- \frac{1}{3 \cdot 5(\omega^2 - 16)}\cos 4t + - \cdots$$

10. The situation is the same as that in Fig. 57 in Sec. 2.11.

12. $y = A_1 \cos t + B_1 \sin t + A_3 \cos 3t + B_3 \sin 3t + \cdots$
where $A_n = -ncb_n/D$, $B_n = (1 - n^2)b_n/D$, $D = (1 - n^2)^2 + n^2c^2$,
$b_1 = 1$, $b_2 = 0$, $b_3 = -\frac{1}{9}$, $b_4 = 0$, $b_5 = \frac{1}{25}, \cdots$.

14. $I = \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt),$

$$A_n = \frac{80(10 - n^2)}{\pi n^2 D_n}, \quad B_n = \frac{800}{n\pi D_n} \quad (n \text{ odd}),$$

$A_n = 0, B_n = 0 \quad (n \text{ even}), \quad D_n = (10 - n^2)^2 + 100\, n^2$; hence

$$\begin{aligned} I = {} & 1.266 \cos t + 1.406 \sin t + 0.003 \cos 3t + 0.094 \sin 3t \\ & - 0.006 \cos 5t + 0.019 \sin 5t - 0.003 \cos 7t + 0.006 \sin 7t \\ & - 0.002 \cos 9t + 0.002 \sin 9t - 0.001 \cos 11t + 0.001 \sin 11t + \cdots \end{aligned}$$

SECTION 10.7. Approximation by Trigonometric Polynomials, page 553

Purpose. We show how to find "best" approximations of a given function by trigonometric polynomials of a given degree N.

Important Concepts

Trigonometric polynomial

Square error

Parseval's identity

Short Courses. Sections 10.5–10.11 can be omitted.

Comment on Quality of Approximation

This quality can be measured in many ways. Particularly important are (i) the absolute value of the maximum deviation over a given interval, and (ii) the mean square error considered here. See Ref. [9] in Appendix 1.

SOLUTIONS TO PROBLEM SET 10.7, page 556

2. $\dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \cdots + \dfrac{1}{N^2}\cos N_x\right)$ if N is odd;

for even N, the last term is $-(4/\pi(N-1)^2)\cos(N-1)x$.

$$E^* = \frac{2\pi^3}{3} - \frac{\pi^3}{2} - \frac{4^2}{\pi}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right);$$ 0.075, 0.075, 0.012, 0.012, 0.0037.

4. $F = \dfrac{\pi^2}{3} - 4\left(\cos x - \dfrac{1}{4}\cos 2x + \dfrac{1}{9}\cos 3x - \cdots + \dfrac{(-1)^{N+1}}{N^2}\cos Nx\right);$

$E^* \approx 4.14, 1.00, 0.38, 0.18, 0.10$

6. $\dfrac{4}{\pi}\left(\sin x - \dfrac{1}{9}\sin 3x + \dfrac{1}{25}\sin 5x - + \cdots\right),$

$$E^* = \frac{\pi^3}{6} - \frac{16}{\pi}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right);$$ 0.075, 0.075, 0.012, 0.012, 0.0037

8. $\dfrac{1}{1^3}\sin x - \dfrac{1}{2^3}\sin 2x + \dfrac{1}{3^3}\sin 3x - + \cdots + \dfrac{(-1)^{N+1}}{N^3}\sin Nx.$

The integral is $3020/945 = 3.196$; $E^* = 0.054, 0.0054, 0.0011, 0.00032, 0.00012$.

10. CAS PROJECT. (a) In part because of the Gibbs phenomenon (see Problem Set 10.3).

(b) For the continuous function (Prob. 4), E^* equals (rounded)

4.14, 1.00, 0.38, 0.18, 0.10, 0.060, 0.039, 0.027, 0.019, 0.014, 0.011, 0.0086, 0.0068, 0.0055, 0.0045, 0.0037, $\cdots$.

For the discontinuous function (Example 1 in the text), E^* equals

8.10, 4.96, 3.57, 2.78, 2.28, 1.93, 1.67, 1.48, 1.32, 1.20, $\cdots$.

It is typical that in the discontinuous case, the Fourier coefficients are only proportional to $1/n$, whereas in continuous cases they are proportional to $1/n^2$ or $1/n^3$, etc.

In Prob. 5 the initial error E^* is very large (863), and E^* decreases very slowly; it is still 1.53 for 800 terms and 0.408 for 3000 terms.

12. $\dfrac{4^2}{\pi^2}\left(1 + \dfrac{1}{9} + \dfrac{1}{25} + \cdots\right) = \dfrac{1}{\pi} \cdot 2\pi$ by Parseval's identity

14. The Fourier series is $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ and (8) gives

$$\frac{1}{2} + \frac{1}{4} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^4 x \, dx.$$

SECTION 10.8. Fourier Integrals, page 557

Purpose. Beginning in this section, we show how ideas from Fourier series can be extended to nonperiodic functions defined on the real line, leading to integrals instead of series.

Main Content, Important Concepts

Fourier integral (5)

Existence Theorem 1

Fourier cosine integral, Fourier sine integral, (10)–(13)

Application to integration

Short Courses. Sections 10.5–10.11 can be omitted.

SOLUTIONS TO PROBLEM SET 10.8, page 563

2. The result suggests to consider $f(x) = \dfrac{\pi}{2}$ if $0 < x < 1$ and $f(x) = 0$ if $x > 1$.

From (10) we obtain $A(w) = \dfrac{\sin w}{w}$, and by inserting this into (11) the result follows.

4. $A(w) = \dfrac{1}{1 - w^2}\cos\dfrac{\pi w}{2}$

6. Taking $f(x) = \pi e^{-x}\cos x \quad (x > 0)$, we obtain from (12)

$$B = 2\int_0^\infty e^{-v} \cos v \sin wv \, dv$$

$$= \int_0^\infty e^{-v} \sin (w + 1)v \, dv + \int_0^\infty e^{-v} \sin (w - 1)v \, dv.$$

Here we used (11) in Appendix A3.1. Integration by parts yields

$$B = \frac{w+1}{1 + (w+1)^2} + \frac{w-1}{1 + (w-1)^2} = \frac{2w^3}{w^4 + 4}.$$

From this and (13) the result follows.

8. $\frac{2}{\pi}\int_0^\infty \left[\left(1 - \frac{2}{w^2}\right) \sin w + \frac{2}{w} \cos w\right] \frac{\cos xw}{w} \, dw$

10. $\frac{4}{\pi}\int_0^\infty (\sin aw - aw \cos aw) \frac{\cos xw}{w^3} \, dw$

12. $\frac{6}{\pi}\int_0^\infty \frac{2 + w^2}{4 + 5w^2 + w^4} \cos xw \, dw$

14. $B(w) = \frac{2}{\pi}\int_0^a v \sin wv \, dv = \frac{2}{\pi w^2} (\sin aw - aw \cos aw)$, so that the *answer* is

$$\frac{2}{\pi}\int_0^\infty \frac{\sin aw - aw \cos aw}{w^2} \sin xw \, dw.$$

16. $\frac{2}{\pi}\int_0^\infty \frac{\pi w - \sin \pi w}{w^2} \sin xw \, dw$

18. $\frac{2}{\pi}\int_0^\infty \frac{w - e^{-1} (w \cos w + \sin w)}{1 + w^2} \sin xw \, dw$

20. PROJECT. (a) Formula (a1): Setting $wa = p$, we have from (11)

$$f(ax) = \int_0^\infty A(w) \cos axw \, dw = \int_0^\infty A\left(\frac{p}{a}\right) \cos xp \frac{dp}{a}.$$

If we again write w instead of p, we obtain (a1).

Formula (a2): From (12) with $f(v)$ replaced by $vf(v)$ we have

$$B^*(w) = \frac{2}{\pi}\int_0^\infty vf(v) \sin wv \, dv = -\frac{dA}{dw}$$

where the last equality follows from (10).

Formula (a3) follows by differentiating (10) twice with respect to w,

$$\frac{d^2A}{dw^2} = -\frac{2}{\pi}\int_0^\infty f^*(v) \cos wv \, dv, \qquad f^*(v) = v^2 f(v).$$

(b) In Prob. 7 we have

$$A = \frac{2}{\pi} w^{-1} \sin w.$$

Hence by differentiating twice,

$$A'' = \frac{2}{\pi}(2w^{-3}\sin w - 2w^{-2}\cos w - w^{-1}\sin w).$$

By (a3) we now get the result, as before,

$$x^2 f(x) = \frac{2}{\pi}\int_0^\infty \left[\left(-\frac{2}{w^3} + \frac{1}{w}\right)\sin w + \frac{2}{w^2}\cos w\right]\cos xw\, dw.$$

(c) $A(w) = (2\sin aw)/\pi w$; see Prob. 7. By differentiation,

$$B^*(w) = -\frac{dA}{dw} = -\frac{2}{\pi}\left(\frac{a\cos aw}{w} - \frac{\sin aw}{w^2}\right).$$

This agrees with the answer to Prob. 14.

(d) The derivation of the following formulas is similar to that of (a1)–(a3).

(d1) $\quad f(bx) = \dfrac{1}{b}\displaystyle\int_0^\infty B\left(\frac{w}{b}\right)\sin xw\, dw \qquad (b > 0)$

(d2) $\quad xf(x) = \displaystyle\int_0^\infty C^*(w)\cos xw\, dw, \qquad C^*(w) = \frac{dB}{dw}, \quad B$ as in (12)

(d3) $\quad x^2 f(x) = \displaystyle\int_0^\infty D^*(w)\sin xw\, dw, \qquad D^*(w) = -\frac{d^2B}{dw^2}.$

SECTION 10.9. Fourier Cosine and Sine Transforms, page 564

Purpose. Fourier cosine and sine transforms are obtained immediately from Fourier cosine and sine integrals, respectively, and we investigate some of their properties.

Content

Fourier cosine and sine transforms

Transforms of derivatives (8), (9)

Comment on Purpose of Transforms

Just as the Laplace transform (Chap. 5), these transforms are designed for solving differential equations. We show this for partial differential equations in Sec. 11.6.

SOLUTIONS TO PROBLEM SET 10.9, page 568

2. From (3) and the answer to Prob. 1 we obtain

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{2\sin w}{w}\cos wx\, dw - \frac{2}{\pi}\int_0^\infty \frac{\sin 2w}{w}\cos wx\, dw.$$

Problem 2 in Problem Set 10.8 shows that the first term is 2 if $0 < x < 1$ and 0 if $x > 1$. Set $2w = u$ in the second term and conclude that the second term is -1 if $0 < x/2 < 1$ or $0 < x < 2$. This agrees with Prob. 1.

6. $f(x) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty e^{-w}\cos wx\, dw = \sqrt{\frac{2}{\pi}}\left(\frac{1}{x^2+1}\right)$ by integration or by Prob. 5.

8. $\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos \frac{1}{2}\pi x}{1 - x^2} \cos xw \, dx = \sqrt{\frac{\pi}{2}} \cos w$ if $|w| < \frac{1}{2}\pi$ and 0 if $|w| > \frac{1}{2}\pi$.

10. $\int_0^\infty \cos wx \, dx = \lim_{x\to\infty} \frac{\sin wx}{w}$ (w fixed!) does not exist. Similarly in (5).

14. $\sqrt{\pi/2}$ if $0 < w < \pi$, and 0 if $w > \pi$

16. $\mathcal{F}_s(xe^{-x^2/2}) = -\mathcal{F}_s((e^{-x^2/2})') = w\mathcal{F}_c(e^{-x^2/2}) = we^{-w^2/2}$

20. **WRITING PROJECT.** Methods include integration, the use of the operational formulas (8) and (9) (independently or together with the use of the tables in Sec. 10.11), and the use of integrals from Sec. 10.8. By presenting this in a systematic fashion, the student should gain a better feeling for these transform methods.

SECTION 10.10. Fourier Transform, page 569

Purpose. Derivation of the Fourier transform from the complex form of the Fourier integral; explanation of its physical meaning and its basic properties.

Main Content, Important Concepts

Complex Fourier integral (4)

Fourier transform (6), its inverse (7)

Spectral representation, spectral density

Transforms of derivatives

Convolution $f * g$

Comments on Content

The complex Fourier integral is relatively easily obtained from the real Fourier integral in Sec. 10.8, and the definition of the Fourier transform is then immediate.

Note that convolution $f * g$ differs from that in Chapter 5, and so does the formula (12) in the convolution theorem (we now have a factor $\sqrt{2\pi}$).

SOLUTIONS TO PROBLEM SET 10.10, page 575

2. $1/((k + iw)\sqrt{2\pi})$

4. $1/((k - iw)\sqrt{2\pi})$

6. $(e^{-iw}(-2i + 2w + iw^2) + 2i)/(w^3\sqrt{2\pi})$

8. $\sqrt{2/\pi}/(1 + w^2)$

10. $\sqrt{2/\pi}(\cos w + w \sin w - 1)/w^2$

12. Let $f(x) = xe^{-x}$ ($x > 0$) and $g(x) = e^{-x}$ ($x > 0$). Then $f' = g - f$ and by (9),

$$iw\mathcal{F}(f) = \mathcal{F}(f') = \mathcal{F}(g) - \mathcal{F}(f);$$

hence

$$(iw + 1)\mathcal{F}(f) = \mathcal{F}(g) = \frac{1}{\sqrt{2\pi}(1 + iw)}.$$

Divide by $1 + iw$ to get the result.

14. $1/((1 + iw)\sqrt{2\pi})$ follows directly from formula 5. For $-\infty < x < 0$ set $x = -t$. Then

$$\int_{-\infty}^0 e^x e^{-iwx} \, dx = \int_0^\infty e^{-t} e^{iwt} \, dt = \frac{e^{-(1-iw)t}}{-(1 - iw)} = \frac{1}{1 - iw}.$$

Together,

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+iw}+\frac{1}{1-iw}\right)=\frac{2}{\sqrt{2\pi}}\cdot\frac{1}{1+w^2}.$$

16. TEAM PROJECT. (a) Use $t = x - a$ as a new variable of integration.

(b) Use $c = 3b$.

(c) Replace w by $w - a$. This gives a new factor e^{iax}.

SOLUTIONS TO CHAPTER 10 REVIEW, page 579

16. $2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots)$

18. $\frac{4k}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \cdots\right)$

20. $2 + \frac{16}{\pi^2}\left(\cos\frac{\pi x}{4} + \frac{1}{9}\cos\frac{3\pi x}{4} + \frac{1}{25}\cos\frac{5\pi x}{4} + \cdots\right)$

22. $\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{1\cdot 3}\cos 2x + \frac{1}{3\cdot 5}\cos 4x + \frac{1}{5\cdot 7}\cos 6x + \cdots\right)$

24. $2(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots)$

26. $\pi - 2(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots)$

28. $\frac{2}{\pi^3}\left((\pi^2 - 6)\sin \pi x - \frac{4\pi^2 - 6}{2^3}\sin 2\pi x + \frac{9\pi^2 - 6}{3^3}\sin 3\pi x\right.$

$$\left. - \frac{16\pi^2 - 6}{4^3}\sin 4\pi x - + \cdots\right)$$

30. For instance, use Prob. 23.

32. For instance, use Prob. 17.

34. 54.403, 4.138, 0.996, 0.376, 0.180, 0.099, 0.060, 0.039, 0.027, 0.019

36. $y = C_1 \cos \omega t + C_2 \sin \omega t$

$$+ \frac{1}{\omega^2 - 1}\sin t - \frac{1}{2^3(\omega^2 - 4)}\sin 2t + \frac{1}{3^3(\omega^2 - 9)}\sin 3t - + \cdots$$

38. $\sqrt{2/\pi}\,(w + \sin w - 2w \cos w)/w^2$

40. $k[(ibw + 1)e^{-ibw} - (iaw + 1)e^{-iaw}]/(w^2\sqrt{2\pi})$

CHAPTER 11 Partial Differential Equations

Change

The Fourier transform method has been included in the section on Fourier integrals for heat problems (Sec. 11.6).

SECTION 11.1 Basic Concepts, page 583

Purpose. To familiarize the student with the following:

Concept of solution, verification of solutions

Superposition principle for homogeneous linear equations

Equations solvable by methods for ordinary differential equations

SOLUTIONS TO PROBLEM SET 11.1, page 584

16. $u = A(y) \cos 3x + B(y) \sin 3x$

18. $u_y/u = -2y, \quad \ln u = -y^2 + \tilde{c}(x), \quad u = c(x) \exp(-y^2)$

20. $u = c_1(x)e^y + c_2(x)e^{-y}$

22. $u_y = q,\ q_y = q,\ q = \tilde{c}(x)e^y,\ u = \int q\, dy = c(x)e^y + h(x)$

24. By the chain rule,

(A) $$yz_x - xz_y = y(z_r r_x + z_\theta \theta_x) - x(z_r r_y + z_\theta \theta_y) = 0.$$

Now $r = (x^2 + y^2)^{1/2}$, $r_x = \frac{1}{2}(x^2 + y^2)^{-1/2}\, 2x = x/r$, $r_y = y/r$, so that $yr_x - xr_y = 0$ in (A) and (A) gives $z_\theta = 0$.

SECTION 11.2. Modeling: Vibrating String, Wave Equation, page 585

Purpose. A careful derivation of the one-dimensional wave equation (more careful than in most other texts, where some of the essential physical assumptions are usually missing).

Short Courses. One may perhaps omit the derivation and just state the wave equation and mention of what c^2 is composed.

SECTION 11.3. Separation of Variables. Use of Fourier Series, page 587

Purpose. This first section in which we solve a "big" problem has several purposes:

1. To familiarize the student with the wave equation and with the typical initial and boundary conditions that physically meaningful solutions must satisfy.
2. To explain and apply the important method of separation of variables, by which the partial differential equation is reduced to ordinary differential equations.
3. To show how Fourier series help to get the final answer, thus seeing the reward of our great and long effort in Chap. 10.
4. To discuss the eigenfunctions of the problem, the basic building blocks of the solution, which lead to a deeper understanding of the whole problem.

Steps of Solution

1. Setting $u = F(x)G(t)$ gives two ordinary differential equations for F and G.
2. The boundary conditions lead to sine and cosine solutions of the latter.
3. A series of those solutions with coefficients determined from the Fourier series of the initial conditions gives the final answer.

SOLUTIONS TO PROBLEM SET 11.3, page 594

2. $0.01 \cos 3t \sin 3x$

4. $u = \dfrac{0.8}{\pi}\left(\cos t \sin x + \dfrac{1}{3^3}\cos 3t \sin 3x + \dfrac{1}{5^3}\cos 5t \sin 5x + \cdots\right)$

6. $u = 2k/(\pi a - a^2)\ (\sin a \cos t \sin x + \frac{1}{4}\sin 2a \cos 2t \sin 2x$

$$+ \tfrac{1}{9}\sin 3a \cos 3t \sin 3x + \cdots)$$

8. $u = \dfrac{4}{5\pi}\left(\dfrac{1}{4}\cos 2t \sin 2x - \dfrac{1}{36}\cos 6t \sin 6x + \dfrac{1}{100}\cos 10t \sin 10x - + \cdots\right)$

10. $u = \sum\limits_{n=1}^{\infty} B_n{}^* \sin nx \sin nt, \quad B_n{}^* = \dfrac{0.04}{\pi n^3}\sin\dfrac{n\pi}{2}$

12. $u = ce^{k(x-y)}$

14. $f'/x^2 f = \dot{g}/y^2 g = 3k, \quad u = ce^{k(x^3+y^3)}$

16. $\dfrac{F''}{F} = -\dfrac{\ddot{G}}{G} = k^2,\ F'' - k^2F = 0,\ \ddot{G} + k^2G = 0$, hence

$$u = (c_1e^{kx} + c_2e^{-kx})(A\cos ky + B\sin ky).$$

Taking the separation constant negative, we obtain a similar result. Taking it zero, we have

$$u = (ax + b)(cy + d).$$

18.

$$u = (A\cos kx + B\sin kx)(C\cos ky + D\sin ky)$$

or

$$u = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ky} + c_4e^{-ky})$$

or

$$u = (ax + b)(cy + d).$$

20. TEAM PROJECT. (c) From the given initial conditions we obtain

$$G_n(0) = B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,$$

$$\dot{G}_n(0) = \lambda_n B_n{}^* + \frac{2A\omega(1 - \cos\omega\pi)}{n\pi(\lambda_n{}^2 - \omega^2)} = 0.$$

(e) $u(0, t) = 0,\ w(0, t) = 0,\ u(L, t) = h(t),\ w(L, t) = h(t)$. The simplest w satisfying these conditions is $w(x, t) = xh(t)/L$. Then

$$v(x, 0) = f(x) - xh(0)/L = f_1(x),$$

$$v_t(x, 0) = g(x) - xh'(0)/L = g_1(x),$$

$$v_{tt} - c^2v_{xx} = -xh''/L.$$

SECTION 11.4. D'Alembert's Solution of the Wave Equation, page 595

Purpose. To show a simpler method of solving the wave equation, which, unfortunately, is not so universal as separation of variables.

Comment on Order of Sections

Section 11.12 on the solution of the wave equation by the Laplace transform may be studied directly after this section. We have placed that material at the end of this chapter because some students may not have studied Chap. 5 on the Laplace transform, which is not a prerequisite for Chap. 11.

Comment on Footnote 2

D'Alembert's *Traité de dynamique* appeared in 1743 and his solution of the vibrating string problem in 1747; the latter makes him, together with Daniel Bernoulli (1700—1782), the founder of the theory of partial differential equations. In 1754 d'Alembert became Secretary of the French Academy of Science and as such the most influential man of science in France.

SOLUTIONS TO PROBLEM SET 11.4, page 597

2. $T = 200$ nt, $\rho = 0.8/(2 \cdot 9.80)$ nt sec^2/meter^2, $c^2 = 4900$ meters^2/sec^2.
Answer: 70 meters/sec

4. $u(0, t) = \frac{1}{2}[f(ct) + f(-ct)] = 0$, $f(-ct) = -f(ct)$, so that f is odd. Also,

$$u(L, t) = \tfrac{1}{2}[f(ct + L) + f(-ct + L)] = 0,$$

hence

$$f(ct + L) = -f(-ct + L) = f(ct - L),$$

which proves the periodicity.

12. $y'^2 - 2y' + 1 = (y' - 1)^2 = 0$, $y = x + c$, $\Psi(x, y) = x - y$, $v = x$, $z = x - y$

14. Hyperbolic, $y'^2 - y' - 2 = (y' + 1)(y' - 2) = 0$, $y + x = c$, $y - 2x = \tilde{c}$,
$v = x + y$, $z = 2x - y$, $u_{vz} = 0$, $u = f_1(x + y) + f_2(2x - y)$

16. Parabolic, $y'^2 + 2y' + 1 = (y' + 1)^2$, $u_{vv} = 0$, $v = x$, $z = x + y$,
$u = vf(z) + \tilde{f}(z) = xf(x + y) + \tilde{f}(x + y)$

18. $u = F(x)G(t)$, $\dfrac{\ddot{G}}{c^2 G} + \dfrac{\gamma^2}{c^2} = \dfrac{F''}{F} = -p^2$, $F_n = \sin \dfrac{n\pi x}{L}$, $\ddot{G} + \lambda_n{}^2 G = 0$,

$\lambda_n{}^2 = \left(\dfrac{cn\pi}{L}\right)^2 + \gamma^2$, etc.

20. TEAM PROJECT. (b) $F_n = \sin(n\pi x/L)$, $G_n = a_n \cos(cn^2\pi^2 t/L^2)$. The solution satisfying the initial conditions is

$$u = \frac{8L^2}{\pi^3}\left(\cos c\left(\frac{\pi}{L}\right)^2 t \sin\frac{\pi x}{L} + \frac{1}{3^3}\cos c\left(\frac{3\pi}{L}\right)^2 t \sin\frac{3\pi x}{L} + \cdots\right)$$

For the string the frequency of the nth normal mode is proportional to n, whereas for the beam it is proportional to n^2.

(c) $u(0, t) = 0$, $u(L, t) = 0$, $u_x(0, t) = 0$, $u_x(L, t) = 0$, Hence

$$F(0) = A + C = 0,\ C = -A,\ F'(0) = \beta(B + D) = 0,\ D = -B.$$

With this we further obtain

$$F(L) = A(\cos\beta L - \cosh\beta L) + B(\sin\beta L - \sinh\beta L) = 0,$$
$$F'(L) = \beta[-A(\sin\beta L + \sinh\beta L) + B(\cos\beta L - \cosh\beta L)] = 0.$$

This homogeneous system has a nontrivial solution if and only if its determinant is zero. Thus $(\cos\beta L - \cosh\beta L)^2 + \sin^2\beta L - \sinh^2\beta L = 0$ or $2 - 2\cos\beta L\cosh\beta L = 0$.

From this we have (17), which can be written

$$\cos\beta L = \frac{1}{\cosh\beta L} \approx 0$$

because $\cosh\beta L$ is very large. This gives approximate solutions

$$\beta L \approx \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \tfrac{7}{2}\pi, \cdots \text{ (more exactly, 4.730, 7.853, 10.996, } \cdots).$$

(d) $F(0) = A + C = 0$, $C = -A$, $F'(0) = \beta(B + D) = 0$, $D = -B$. Then

$$F(x) = A(\cos\beta x - \cosh\beta x) + B(\sin\beta x - \sinh\beta x),$$
$$F''(L) = \beta^2[-A(\cos\beta L + \cosh\beta L) - B(\sin\beta L + \sinh\beta L)] = 0,$$
$$F'''(L) = \beta^3[A(\sin\beta L - \sinh\beta L) - B(\cos\beta L + \cosh\beta L)] = 0.$$

The determinant $(\cos\beta L + \cosh\beta L)^2 + \sin^2\beta L - \sinh^2\beta L$ of this system must be zero, and from this the result follows.

From (18) we have

$$\cos\beta L = \frac{-1}{\cosh\beta L} \approx 0$$

because $\cosh\beta L$ is very large. This gives the approximate solutions

$$\beta L \approx \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \cdots \text{ (more exactly, 1.875, 4.694, 7.855, } \cdots).$$

SECTION 11.5. Heat Equation: Solution by Fourier Series, page 600

Purpose. This section has two purposes:

1. To solve a typical heat problem by steps similar to those for the wave equation, pointing to the two main differences: only *one* initial condition (instead of two) and u_t (instead of u_{tt}), resulting in exponential functions in t (instead of cosine and sine in the wave equation).
2. Solution of Laplace's equation (which can be interpreted as a time-independent heat equation in two dimensions).

Comments on Content

Additional points to emphasize are

More rapid decay with increasing n,

Difference in time evolution in Figs. 267 and 263,

Zero can be an eigenvalue (see Example 4),

Three standard types of boundary value problems,

Analogy of electrostatic and (steady-state) heat problems.

Problem Set 11.5 includes additional heat problems and types of boundary conditions.

SOLUTIONS TO PROBLEM SET 11.5, page 608

2. $\lambda_1^2 = (\ln 2)/20$, $c^2 = (L^2/\pi^2)(\ln 2)/20 = 0.0035L^2$

4. $u = k \sin 0.2\pi x\, e^{-1.752\pi^2 t/25}$

6. $u = \dfrac{16}{\pi^2}\left(\sin 0.1\pi x\, e^{-0.01752\pi^2 t} - \dfrac{1}{9}\sin 0.3\pi x\, e^{-0.01752(3\pi)^2 t} + - \cdots\right)$

8. $u(x, t) = u_I(x) + u_{II}(x, t)$ with u_I as in Prob. 7 and

$$u_{II} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}\, e^{-(cn\pi/L)^2 t},$$

where

$$B_n = \frac{2}{L}\int_0^L [f(x) - u_I(x)] \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx + \frac{2}{n\pi}[(-1)^n U_2 - U_1].$$

10. $u = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\cos x\, e^{-t} + \dfrac{1}{9}\cos 3x\, e^{-9t} + \dfrac{1}{25}\cos 5x\, e^{-25t} + \cdots\right)$

12. $u(x, t) = \cos 2x\, e^{-4t}$

14. $w'' = -Ne^{-\alpha x}/c^2$; make $w(0) = w(L) = 0$ to get the function

$$w(x) = \frac{N}{c^2\alpha^2}\left[-e^{-\alpha x} - \frac{1}{L}(1 - e^{-\alpha L})x + 1\right].$$

16. $-\dfrac{K\pi}{L}\displaystyle\sum_{n=1}^{\infty} nB_n\, e^{-\lambda_n^2 t}$

18. CAS PROJECT. **(a)** $u = \sin \pi x \sinh \pi y/\sinh 2\pi$.

(b) $u_y(x, 0, t) = u_y(x, 2, t) = 0$, $u = \sin m\pi x \cos n\pi y$.

20. $u = F(x)G(y)$, $F = A\cos px + B\sin px$, $u_x(0, y) = F'(0)G(y) = 0$, $B = 0$, $G = C\cosh py + D\sinh py$, $u_y(x, b) = F(x)G'(b) = 0$, $C = \cosh pb$, $D = -\sinh pb$, $G = \cosh(pb - py)$. For $u = \cos px \cosh p(b - y)$ we get

$$u_x(a, y) + hu(a, y) = (-p\sin pa + h\cos pa)\cosh p(b - y) = 0.$$

Hence p must satisfy $\tan ap = h/p$, which has infinitely many positive real solutions $p = \gamma_1, \gamma_2, \cdots$, as you can illustrate by a simple sketch. *Answer:*

$$u = u_n = \cos \gamma_n x \cos \gamma_n(b - y),$$

where $\gamma = \gamma_n$ satisfies $\gamma \tan \gamma a = h$.

To determine coefficients of series of u_n's from a boundary condition at the lower side is difficult because that would not be a Fourier series, the γ_n's being only approximately regularly spaced. See [C1], pp. 114–119, 167.

SECTION 11.6. Heat Equation: Solution by Fourier Integrals and Transforms, page 610

Purpose. Whereas we solved the problem of a finite bar in the last section by using Fourier series, we show that for an infinite bar (practically, a long insulated wire) we can use the Fourier integral for the same purpose. Figure 271 shows the time evolution for a "rectangular" initial temperature (100°C between $x = -1$ and $+1$, zero elsewhere), giving bell-shaped curves as for the density of the normal distribution.

We also show typical applications of the Fourier transform and the Fourier sine transform to the heat equation.

Short Courses. This section can be omitted.

SOLUTIONS TO PROBLEM SET 11.6, page 615

2. $A(p) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{1}{1+v^2}\cos pv\,dv = \frac{2}{\pi}\cdot\frac{\pi}{2}e^{-p} = e^{-p}$ by (15), Sec. 10.8,

and
$$u(x, t) = \int_0^\infty e^{-(p+c^2p^2t)}\cos px\,dp.$$

4. $A(p) = \dfrac{2}{\pi(1+p^2)}$, $B(p) = 0$, $u = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{1}{1+p^2}\cos px\, e^{-c^2p^2t}\,dp$

10. CAS PROJECT. **(a)** Set $w = -v$ in (21) to get $\operatorname{erf}(-x) = -\operatorname{erf} x$.

(b) See (36) in Appendix A3.1.

(e) $u(x, t) = \dfrac{U_0}{2}\left[\operatorname{erf}\dfrac{1-x}{2c\sqrt{t}} + \operatorname{erf}\dfrac{1+x}{2c\sqrt{t}}\right]$, $(t > 0)$

(f) $u(x, t) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}(x/2c\sqrt{t})$

SECTION 11.7. Modeling: Membrane, Two-Dimensional Wave Equation, page 616

Purpose. A careful derivation of the two-dimensional wave equation governing the motions of a drumhead, from physical assumptions (the analog of the modeling in Sec. 11.2).

SECTION 11.8. Rectangular Membrane. Use of Double Fourier Series, page 619

Purpose. To solve the two-dimensional wave equation in a rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$ ("rectangular membrane") by separation of variables and double Fourier series.

Comment on Content

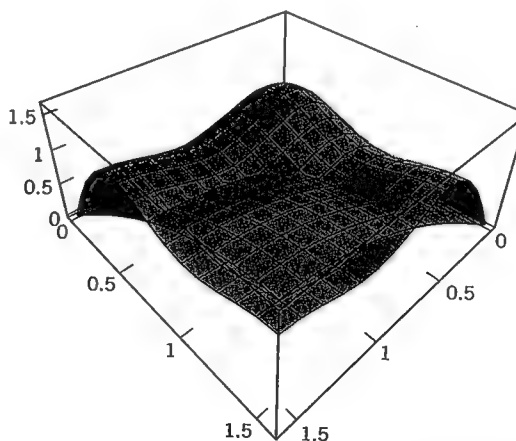
New features as compared to the one-dimensional case (Sec. 11.3) are as follows:

1. We have to separate twice, first by $u = F(x, y)G(t)$, then the Helmholtz equation for F by $F = H(x)Q(y)$.
2. We get a double sequence of infinitely many eigenvalues λ_{mn} and eigenfunctions u_{mn}; see (12), (13).
3. We need double Fourier series (easily obtainable from the usual Fourier series) to get a solution that also satisfies the initial conditions.

SOLUTIONS TO PROBLEM SET 11.8, page 626

6. $B_{mn} = 16/mn\pi^2$ (m and n odd), 0 otherwise

8. $B_{mn} = (-1)^{m+n}4/mn$

10. CAS PROJECT. (b) The figure shows the first partial sum (a single term) and the partial sum of the terms up to that with coefficient b_{55} (9 terms).

Section 11.8. CAS Project 10(b). Two partial sums

12. $u = k \cos \pi\sqrt{2}t \sin \pi x \sin \pi y$

14. $\displaystyle \frac{64k}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m(m^2-4)n(n^2-4)} \cos(\pi t\sqrt{m^2+n^2}) \sin m\pi x \sin n\pi y$ (m, n odd)

16. $A = ab$, $b = A/a$, so that from (12) with $m = n = 1$ by differentiating with respect to a and equating the derivative to zero, we obtain

$$\left(\frac{\lambda_{11}{}^2}{c^2\pi^2}\right)' = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)' = \left(\frac{1}{a^2} + \frac{a^2}{A^2}\right)' = \frac{-2}{a^3} + \frac{2a}{A^2} = 0;$$

hence $a^4 = A^2$, $a^2 = A$, $b = A/a = a$.

18. $c\pi\sqrt{260}$ (corresponding eigenfunctions $F_{4,16}$, $F_{16,14}$), etc.

20. $\displaystyle \frac{64a^2b^2}{\pi^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^3n^3} \cos\left(\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\, t\right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$ (m, n odd)

SECTION 11.9. Laplacian in Polar Coordinates, page 626

Purpose. A detailed discussion of the transformation of the Laplacian into polar coordinates as a typical case of a task often required in applications. The result (4) will be needed in the next section.

Short Courses. This section can be omitted.

SOLUTIONS TO PROBLEM SET 11.9, page 628

6. TEAM PROJECT. (a) $r^2 \cos 2\theta = r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2$, $r^2 \sin 2\theta = 2xy$, etc.

(c) $\displaystyle u = \frac{400}{\pi}\left(r \sin\theta + \frac{1}{3} r^3 \sin 3\theta + \frac{1}{5} r^5 \sin 5\theta + \cdots\right)$

(d) The form of the series results as in (b) and the formulas for the coefficients follow from

$$u_r(R, \theta) = \sum_{n=1}^{\infty} nR^{n-1}(A_n \cos n\theta + B_n \sin n\theta) = f(\theta).$$

8. $u(\theta) = 10(\frac{1}{2} + \frac{1}{2}\cos 2\theta)$. *Answer:* $5 + 5r^2 \cos 2\theta$

10. $u = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(r\cos\theta + \dfrac{1}{9}r^3\cos 3\theta + \dfrac{1}{25}r^5\cos 5\theta + \cdots\right)$

12. $u = 75r\sin\theta - 25r^3\sin 3\theta$. Note that this also follows from Prob. 7 because of the skew symmetry of the boundary condition as a function of θ.

14. $u = \dfrac{80}{\pi}\left(r\sin\theta + \dfrac{1}{3^3}r^3\sin 3\theta + \dfrac{1}{5^3}r^5\sin 5\theta + \cdots\right)$

SECTION 11.10. Circular Membrane: Use of Fourier-Bessel Series, page 629

Purpose. To derive the function that gives the vibrations of a circular membrane, by solving the wave equation in polar coordinates.

Comment on Content

We concentrate on the simpler case of radially symmetric vibrations, that is, vibrations independent of the angle. (For eigenfunctions depending on the angle, see Probs. 11–18.) We do three steps:

1. $u = W(r)G(t)$ gives for W Bessel's equation with $\nu = 0$, hence solutions $W(r) = J_0(kr)$.
2. We satisfy the boundary condition $W(R) = 0$ by choosing suitable values of k.
3. A Fourier–Bessel series (13) helps to get the solution (12) of the entire problem.

Short Courses. This section can be omitted.

SOLUTIONS TO PROBLEM SET 11.10, page 634

2. $f_1 = ck_1/2\pi = c\alpha_1/2\pi R = 0.3827c/R = 0.3827\sqrt{T/\rho R^2}$

4. In polar coordinates the boundary has the simple representation $R = const.$

6. **CAS PROJECT (b)** Error 0.04864 ($m = 1$), 0.02229, 0.01435, 0.01056, 0.00835, 0.00690, 0.00589, 0.00512, 0.00454, 0.00408 ($m = 10$)

8. From (24), Sec. 4.5, we have $(rJ_1(r))' = rJ_0(r)$. By integration

$$\text{(A)} \qquad \int_0^{\alpha_m} rJ_0(r)\,dr = \alpha_m J_1(\alpha_m).$$

$a_m = 0$ because the initial deflection is zero. From (15) and (A), with $g(r) = 1$ and $\alpha_m r = s$, we obtain

$$b_m = \frac{2}{\alpha_m J_1^2(\alpha_m)}\int_0^1 rJ_0(\alpha_m r)\,dr$$

$$= \frac{2}{\alpha_m J_1^2(\alpha_m)}\int_0^{\alpha_m} \frac{s}{\alpha_m} J_0(s)\,\frac{ds}{\alpha_m}$$

$$= \frac{2}{\alpha_m{}^3 J_1{}^2(\alpha_m)} \alpha_m J_1(\alpha_m).$$

Hence the series is

$$u(r, t) = 2 \sum_{m=1}^{\infty} \frac{1}{\alpha_m{}^2 J_1(\alpha_m)} \sin \alpha_m t \; J_0(\alpha_m r).$$

10. $f(0) = 1$; 1.10801, 0.96823, 1.01371, 0.99272, 1.00436, and we see that the last value is already correct to 3 significant digits.

18. $\alpha_{11}/2\pi \approx 0.6099$ (see Table A1 in Appendix 5)

SECTION 11.11. Laplace's Equation in Cylindrical and Spherical Coordinates. Potential, page 636

Purpose. 1. Transformation of the Laplacian into cylindrical coordinates (which is trivial because of Sec. 11.9) and spherical coordinates; some remarks on areas in which Laplace's equation is basic.

2. Separation of the Laplace equation in spherical coordinates and application to a typical boundary value problem. For simplicity we consider a boundary value problem for a sphere with boundary values depending only on ϕ. We do three steps:

1. $u = G(r)H(\phi)$ and separation gives for H Legendre's equation.
2. Continuity requirements restrict H to Legendre polynomials.
3. A Fourier–Legendre series (17) helps to get the solution (16) of the interior problem. Similarly for the exterior problem, whose solution is (19).

Short Courses. Omit the derivation of the Laplacian in cylindrical and spherical coordinates.

SOLUTIONS TO PROBLEM SET 11.11, page 641

4. $u = -80 \ln r/(\ln 2) + 300$

6. For $u = u(r)$ we get from (7)

$$\nabla^2 u = u'' + \frac{2}{r} u' = 0, \qquad \frac{u''}{u'} = -\frac{2}{r}, \qquad \ln u' = -2 \ln r + c_1,$$

$$u' = \frac{\tilde{c}}{r^2}, \qquad u = \frac{c}{r} + k.$$

10. $\tilde{f}(w) = w, \quad A_n = \dfrac{2n+1}{2} \displaystyle\int_{-1}^{1} w P_n(w)\, dw$. Since $w = P_1(w)$ and the $P_n(w)$ are orthogonal on the interval $-1 \leqq w \leqq 1$, we obtain $A_1 = 1$, $A_n = 0$ $(n > 1)$. *Answer:* $u = r \cos \phi$. Of course, this is at once seen by inspection.

12. $u = -\frac{2}{3} r^2 P_2(\cos \phi) + \frac{2}{3} = r^2(-\cos^2 \phi + \frac{1}{3}) + \frac{2}{3}$

14. $u = 4r^3 P_3(\cos \phi) - 2r^2 P_2(\cos \phi) + r P_1(\cos \phi) - 2$

16. $f(\phi) = \cos \phi$, $u_{\text{int}} = r \cos \phi$, $u_{\text{ext}} = r^{-2} \cos \phi$, $f(\phi) = \cos 2\phi = 2 \cos^2 \phi - 1$, $2x^2 - 1 = \frac{4}{3} P_2(x) - \frac{1}{3}$, $u_{\text{int}} = \frac{4}{3} r^2 P_2(\cos \phi) - \frac{1}{3}$, $u_{\text{ext}} = \dfrac{4}{3r^3} P_2(\cos \phi) - \dfrac{1}{3r}$

18. $A_4 = 0$, $A_5 = 605/16$, $A_6 = 0$, $A_7 = -4125/128$, $A_8 = 0$, $A_9 = 7315/256$, $A_{10} = 0$

20. Set $\frac{1}{r} = \rho$ and consider $u(\rho, \theta, \phi) = rv(r, \theta, \phi)$. By differentiation,

$$u_\rho = (v + rv_r)\left(-\frac{1}{\rho^2}\right), \quad u_{\rho\rho} = (2v_r + rv_{rr})\frac{1}{\rho^4} + \frac{2}{\rho^3}(v + rv_r).$$

Thus

$$u_{\rho\rho} + \frac{2}{\rho}u_\rho = \frac{1}{\rho^4}(2v_r + rv_{rr}) = r^5\left(v_{rr} + \frac{2}{r}v_r\right).$$

By substituting this and $u_{\phi\phi} = rv_{\phi\phi}$, etc., into (7) [written in terms of ρ] and dividing by r^5 we obtain the result.

22. $v = r^{-2}\cos\theta\sin\theta = xy/(x^2 + y^2)^2$

24. TEAM PROJECT. (a) The two drops over a portion of the cable of length Δx are $-Ri\Delta x$ and $-L(\partial i/\partial t)\Delta x$, respectively. Their sum equals the difference $u_{x+\Delta x} - u_x$. Divide by Δx and let $\Delta x \to 0$.

(c) To get the first equation, differentiate the first transmission line equation with respect to x and use the second equation to replace i_x and i_{xt}:

$$\begin{aligned} -u_{xx} &= Ri_x + Li_{xt} \\ &= R(-Gu - Cu_t) + L(-Gu_t - Cu_{tt}). \end{aligned}$$

Now collect terms. Similarly for the second equation.

(d) Set $\frac{1}{RC} = c^2$. Then $u_t = c^2u_{xx}$, the heat equation. By (10), (11), Sec. 11.5,

$$u = \frac{4U_0}{\pi}\left(\sin\frac{\pi x}{l}e^{-\lambda_1^2 t} + \frac{1}{3}\sin\frac{3\pi x}{l}e^{-\lambda_3^2 t} + \cdots\right), \qquad \lambda_n^2 = \frac{n^2\pi^2}{l^2RC}.$$

(e) $u = U_0\cos(\pi t/l\sqrt{LC})\sin(\pi x/l)$

SECTION 11.12. Solution by Laplace Transforms, page 643

Purpose. For students familiar with Chap. 5 we show that the Laplace transform also applies to certain partial differential equations, where the subsidiary equation must be expected to be an ordinary differential equation.

Short Courses. This section can be omitted.

SOLUTIONS TO PROBLEM SET 11.12, page 646

2. Use $c = \sqrt{T/\rho}$.

4. $u(x, t) = t + 1 - (t - x^2)u(t - x^2) = \begin{cases} t + 1 & \text{if } t \leqq x^2 \\ x^2 + 1 & \text{if } t \geqq x^2 \end{cases}$

(where $u(t - x^2)$ is the unit step fuction) as obtained from

$$U(x, s) = \frac{1 + s}{s^2} + c(s)e^{-sx^2}$$

with $c(s) = -1/s^2$ as obtained from $u(0, t) = 1$, $U(0, s) = 1/s$.

6. $u = f(x)g(t), \quad xf'g + f\dot{g} = xt$, hence

$$x\frac{f'}{f} + \frac{\dot{g}}{g} = \frac{xt}{fg}.$$

To complete the separation, we take $f(x) = x$, obtaining

$$1 + \frac{\dot{g}}{g} = \frac{t}{g}, \qquad \dot{g} + g = t, \qquad g = ce^{-t} + t - 1;$$

hence

$$u = x(ce^{-t} + t - 1),$$

which satisfies $u(0, t) = 0$. Also, $u(x, 0) = x(c - 1)$. Thus $c = 1$ and the answer is, as before,

$$u = x(e^{-t} + t - 1).$$

8. From $W = F(s)e^{-(x/c)\sqrt{s}}$ and the convolution theorem we have

$$w = f * \mathcal{L}^{-1}\{e^{-k\sqrt{s}}\}, \qquad k = \frac{x}{c}.$$

From this and formula 39 in Sec. 5.9 we get, as asserted,

$$w = \int_0^t f(t - \tau)\frac{k}{2\sqrt{\pi\tau^3}}e^{-k^2/4\tau}\,d\tau.$$

10. $W_0(x, s) = s^{-1}e^{-\sqrt{s}x/c}$, $\mathcal{L}\{u(t)\} = 1/s$, and since $w(x, 0) = 0$,

$$W(x, s) = F(s)sW_0(x, s) = F(s)[sW_0(x, s) - w(x, 0)]$$

$$= F(s)\,\mathcal{L}\left\{\frac{\partial w_0}{\partial t}\right\}.$$

Now apply the convolution theorem.

SOLUTIONS TO CHAPTER 11 REVIEW, page 647

22. $u = c(x)e^{-y} - \frac{1}{2}x^2$

24. $u = g(y)(1 - e^{-x}) + f(y)$

26. $u = (Ae^{kx} + Be^{-kx})(C\cos ky + D\sin ky)$,
$u = (ax + b)(cy + d)$,
$u = (A\cos kx + B\sin kx)(Ce^{ky} + De^{-ky})$

28. Parabolic, $y'^2 - 6y' + 9 = (y' - 3)^2$, $v = x$, $z = y - 3x$,
$u = xf_1(y - 3x) + f_2(y - 3x)$

30. $\frac{3}{4}\cos 2t\sin x - \frac{1}{4}\cos 6t\sin 3x$

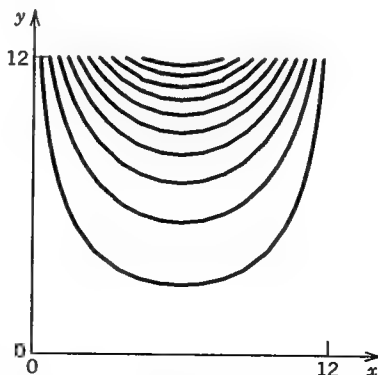
32. $u = \dfrac{4}{\pi}\left(\cos 2t\sin x - \dfrac{1}{9}\cos 6t\sin 3x + \dfrac{1}{25}\cos 10t\sin 5x - + \cdots\right)$

34. $u = \dfrac{400}{\pi^2}\left(\sin\dfrac{\pi x}{100}e^{-0.001143t} - \dfrac{1}{9}\sin\dfrac{3\pi x}{100}e^{-0.010286t} + - \cdots\right)$

36. $u = 5\pi^2 - 60\left(\cos x\,e^{-t} - \dfrac{1}{4}\cos 2x\,e^{-4t} + \dfrac{1}{9}\cos 3x\,e^{-9t} - + \cdots\right)$

38. $u = \pi - \dfrac{32}{\pi}\left(\dfrac{1}{4}\cos 2x\,e^{-4t} + \dfrac{1}{36}\cos 6x\,e^{-36t} + \cdots\right)$

40. $u = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{12} \frac{\sinh [(2n-1)\pi y/12]}{\sinh (2n-1)\pi}$

42.

Chapter 11 Review. Equipotential lines in Prob. 42

44. $\lambda_{11}/2\pi = c\pi(\sqrt{1+1})/2\pi = 1/\sqrt{2}$

46. Area $\pi R^2/2 = 1$, $R = \sqrt{2/\pi}$,
$ck_{11}/2\pi = k_{11}/2\pi = \alpha_{11}/2\pi R = 3.832/2\pi\sqrt{2/\pi} = 3.832/\sqrt{8\pi}$

48. $u = \frac{(u_0 - u_1)r_0 r_1}{(r_1 - r_0)r} + \frac{u_1 r_1 - u_0 r_0}{r_1 - r_0}$, where r is the distance from the center of the spheres

50. $f(\phi) = 4\cos^3 \phi$. Now, by (11′), Sec. 4.3,

$$\cos^3 \phi = \tfrac{2}{5}P_3(\cos \phi) + \tfrac{3}{5}P_1(\cos \phi).$$

Answer:

$$u = \tfrac{8}{5}r^3 P_3(\cos \phi) + \tfrac{12}{5}rP_1(\cos \phi).$$

PART D. COMPLEX ANALYSIS

CHAPTER 12 Complex Numbers and Functions. Conformal Mapping

Major Changes

The old chapter on conformal mapping has been absorbed into Chap. 12, beginning with a general introduction to conformality in Sec. 12.5, continuing with the conformal mapping of the elementary complex functions in Secs. 12.6–12.8 and concluding with a special section on linear fractional transformations (Sec. 12.9). This gives a better understanding of those functions because we now discuss their geometric properties (their mapping properties) simultaneously with their analytic formulas, as we do it all the time in calculus.

SECTION 12.1. Complex Numbers. Complex Plane, page 652

Purpose. To discuss the algebraic operations for complex numbers and the representation of complex numbers as points in the plane.

Main Content, Important Concepts

Complex number, real part, imaginary part, imaginary unit

The four algebraic operations in complex

Complex plane, real axis, imaginary axis

Complex conjugates

Two Suggestions on Content

1. Of course, at the expense of a small conceptual concession, one can also start immediately from (4), (5),

$$z = x + iy, \qquad i^2 = -1$$

and go on from there.

2. If students have some knowledge of complex numbers, the practical division rule (7) and perhaps (8) and (9) may be the only items to be recalled in this section. (But I personally would do ten minutes more in any case.)

SOLUTIONS TO PROBLEM SET 12.1, page 656

2. $iz = -2 + 4i,\ -1 - i,\ 2 + 5i$

4. $-96 + 280i$ **6.** $-7 - 26i$ **8.** $-0.1 + 1.3i$

10. $(7 - 24i)/625,\ (7 + 24i)/625$ **12.** $29/25$

14. $4x^3y - 4xy^3,\ 4x^2y^2$ **16.** $(x^2 - y^2)/(x^2 + y^2)$

20. $z_1 z_2 = 0$ if and only if

$$x_2 x_1 - y_2 y_1 = 0$$
$$y_2 x_1 + x_2 y_1 = 0.$$

Let $z_2 \neq 0$, so that $x_2^2 + y_2^2 \neq 0$, the coefficient determinant of our homogeneous system of equations in the "unknowns" x_1 and y_1, which therefore must be zero; hence $z_1 = 0$.

SECTION 12.2. Polar Form of Complex Numbers. Powers and Roots, page 657

Purpose. To give the student a firm grasp of the polar form, including the principal value Arg z, and its application in multiplication and division.

Main Content, Important Concepts

Absolute value $|z|$, argument θ, principal value Arg θ

Triangle inequality

Multiplication and division in polar form

nth root, nth roots of unity (16)

SOLUTIONS TO PROBLEM SET 12.2, page 662

2. $\sqrt{8}(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi)$ **4.** $10(\cos \pi + i \sin \pi)$ **6.** $\cos \frac{1}{2}\pi - i \sin \frac{1}{2}\pi$

8. $(1/\sqrt{18})(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$ **10.** $\sqrt{37/8}(\cos 2.19105 - i \sin 2.19105)$

12. π, -3.0419 **14.** $3\pi/4$

16. i **18.** $3 + \sqrt{27}\, i$

20. TEAM PROJECT. **(a)** Use (15).

(b) Use those formulas (10) in the form

$$\cos \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(1 + \cos \theta)}, \qquad \sin \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(1 - \cos \theta)},$$

multiply them by $\sqrt{r}$,

$$\sqrt{r} \cos \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(r + r\cos \theta)}, \qquad \sqrt{r} \sin \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(r - r\cos \theta)},$$

use $r \cos \theta = x$, and finally choose the sign of Im $\sqrt{z}$ in such a way that sign $[(\text{Re}\sqrt{z})(\text{Im}\sqrt{z})] = \text{sign}\, y$.

(c) $\pm\sqrt{2}(1 + i)$, $6 + 4i$, $5 - \sqrt{2}\, i$

22. $2(\cos \frac{1}{6}k\pi + i \sin \frac{1}{6}k\pi)$, $k = 1, 5, 9$, that is, $\sqrt{3} + i$, $-\sqrt{3} + i$, $-2i$.

24. $\pm(1 \pm i)$

26. $\pm(2 + i)$, $\pm(1 - 2i)$ is obtained by taking the square root of each of the two solutions in Prob. 25.

28. Using (18) in Team Project 20, we obtain

$$z = \tfrac{1}{2}(5 + i) \pm \sqrt{\tfrac{1}{4}(5 + i)^2 - 8 - i} = \tfrac{1}{2}(5 + i) \pm \sqrt{-2 + \tfrac{5}{2}i}$$
$$= \tfrac{1}{2}(5 + i) \pm \left[\sqrt{\tfrac{1}{2}(\tfrac{5}{2} + (-2))} + i\sqrt{\tfrac{1}{2}(\tfrac{5}{2} - (-2))}\right]$$
$$= \tfrac{1}{2}(5 + i) \pm \left[\tfrac{1}{2} + \tfrac{3}{2}i\right] = \begin{cases} 3 + 2i \\ 2 - i. \end{cases}$$

30. Quadratic equation in z^2 with solutions

$$z^2 = 3(\tfrac{1}{2} + i) \pm (1.5 + i) = 3 + 4i \text{ and } 2i,$$

with the roots evaluated by (18). From this, by (18), we get the four solutions

$$\pm\sqrt{3 + 4i} = \pm[2 + i], \qquad \pm\sqrt{2i} = \pm[1 + i].$$

34. $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$,

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= 2(x_1^2 + x_2^2 + y_1^2 + y_2^2) = 2(|z_1|^2 + |z_2|^2).$$

The name results from the fact that the equality relates the lengths of the sides and the diagonals of a parallelogram with sides determined by the vectors corresponding to z_1 and z_2. For the importance of the equality in Functional Analysis, see Ref. [9] listed in Appendix 1.

36. $|z| = \sqrt{x^2 + y^2} \geqq |x|$, etc.

SECTION 12.3. Derivative. Analytic Function, page 663

Purpose. To define (complex) analytic functions—the class of functions complex analysis is concerned with—and the concepts needed for that definition, in particular, derivatives.

This is preceded by a collection of a few standard concepts on sets in the complex plane that we shall need from time to time in the chapters on complex analysis.

Main Content, Important Concepts

Unit circle, unit disk, open and closed disks

Domain, region

Complex function

Limit, continuity

Derivative

Analytic function

Comment on Content

The most important concept in this section is that of an **analytic function.** The other concepts resemble those of real calculus. The most important new *idea* is connected with the **limit:** the approach in infinitely many possible directions. This yields the negative result in Example 4 and—much more importantly—the **Cauchy–Riemann equations** in the next section.

SOLUTIONS TO PROBLEM SET 12.3, page 668

2. Annulus with center $4 - 2i$ bounded by the circles of radius $\frac{1}{2}$ and 2

4. Disk without its center $1 + i$, radius $\sqrt{2}$. Such domains will be crucial in connection with residue integration in Chap. 15.

6. We obtain

$$\frac{x}{x^2 + y^2} < 1, \qquad x < x^2 + y^2, \qquad \tfrac{1}{4} < (x - \tfrac{1}{2})^2 + y^2,$$

the exterior of the circle of radius $\frac{1}{2}$ with center at $\frac{1}{2}$.

8. Angular region $-\frac{1}{4}\pi < \arg z < \frac{1}{4}\pi$

10. 4, −4 **12.** 0.6, 0.8

14. $(r^2 \cos 2\theta)/r \to 0$ as $r \to 0$; yes

16. $(r \cos\theta - r \sin\theta)/r^2 = (\cos\theta - \sin\theta)/r$; no

18. 625000 **20.** $-110 + 70i$ **22.** 0

24. TEAM PROJECT. (a) Use $\operatorname{Re} f(z) = [f(z) + \overline{f(z)}]/2$, $\operatorname{Im} f(z) = [f(z) - \overline{f(z)}]/2i$.

(b) Assume that $\lim_{z\to z_0} f(z) = l_1$, $\lim_{z\to z_0} f(z) = l_2$, $l_1 \neq l_2$. For every $\epsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(z) - l_j| < \epsilon \quad \text{when} \quad 0 < |z - z_0| < \delta_j, \quad j = 1, 2.$$

Hence for $\epsilon = |l_1 - l_2|/2$ and $0 < |z - z_0| < \delta$, where $\delta \leqq \delta_1$, $\delta \leqq \delta_2$, we have

$$\begin{aligned} |l_1 - l_2| &= |[f(z) - l_2] - [f(z) - l_1]| \\ &\leqq |f(z) - l_2| + |f(z) - l_1| < 2\epsilon = |l_1 - l_2|. \end{aligned}$$

(c) By continuity, for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(z) - f(a)| < \epsilon$ when $|z - a| < \delta$. Now $|z_n - a| < \delta$ for all sufficiently large n since $\lim z_n = a$. Thus $|f(z_n) - f(a)| < \epsilon$ for these n.

(d) The proof is as in calculus. We write

$$\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = \eta.$$

Then from the definition of a limit it follows that for any given $\epsilon > 0$ there is a $\delta > 0$ such that $|\eta| < \epsilon$ when $|z - z_0| < \delta$. From this and the triangle inequality,

$$|f(z) - f(z_0)| = |z - z_0||f'(z_0) + \eta| \leqq |z - z_0||f'(z_0)| + |z - z_0|\epsilon,$$

which approaches 0 as $|z - z_0| \to 0$.

(e) The quotient in (4) is $\Delta x/\Delta z$, which is 0 if $\Delta x = 0$ but 1 if $\Delta y = 0$, so that it has no limit as $\Delta z \to 0$.

(f) $$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{(z + \Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z} = z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}.$$

When $z = 0$ the expression on the right approaches zero as $\Delta z \to 0$. When $z \neq 0$ and $\Delta z = \Delta x$, then $\overline{\Delta z} = \Delta x$ and that expression approaches $z + \bar{z}$. When $z \neq 0$ and $\Delta z = i\Delta y$, then $\overline{\Delta z} = -i\Delta y$ and that expression approaches $-z + \bar{z}$. This proves the statement.

SECTION 12.4. Cauchy–Riemann Equations. Laplace's Equation, page 669

Purpose. To derive and explain the most important equations in this chapter, the Cauchy–Riemann equations, which constitute the basic criterion for analyticity.

Main Content, Important Concepts

Cauchy–Riemann equations (1)

These equations as a criterion for analyticity (Theorems 1 and 2)

Derivative in terms of partial derivatives, (4), (5)

Relation of analytic functions to Laplace's equation

Harmonic function, conjugate harmonic

Comment on Content

(4), (5), and Example 3 will be needed occasionally.

The relation to Laplace's equation is basic, as mentioned in the text.

SOLUTIONS TO PROBLEM SET 12.4, page 673

2. No **4.** Yes ($z \neq 0$) **6.** Yes ($z \neq 0$) **8.** Yes ($z \neq \pm 1, \pm i$) **10.** No

12. No. Note that this is $x^2 - y^2 - 2xyi = (x - iy)^2 = \bar{z}^2$.

14. TEAM PROJECT. (a) $u = const$, $u_x = u_y = 0$, $v_x = v_y = 0$ by (1), $v = const$, $f = u + iv = const$.

(b) Same idea as in (a).

(c) $f' = u_x + iv_x = 0$ by (4). Hence $v_y = 0$, $u_y = 0$ by (1), $f = u + iv = const$.

18. No **20.** $\ln |z| + i \operatorname{Arg} z$ **22.** $\sin x \cosh y + i \cos x \sinh y$

24. No **26.** $a = 0$, $\frac{1}{2}b(y^2 - x^2)$ **28.** $a = 2$, $-\sin 2x \sinh 2y$

30. Students should observe the orthogonality of the two families, which will be discussed in the next section, as a consequence of conformality.

SECTION 12.5. Geometry of Analytic Functions: Conformal Mapping, page 674

Purpose. To show conformality (preservation of angles in size and sense) of the mapping by an analytic function $w = f(z)$; exceptional are points z at which $f'(z) = 0$.

Main Content, Important Concepts

Concept of mapping

Complex functions as mappings

Definition of conformality

Critical point

Conformality (Theorem 1)

Comment on the Proof

The crucial point is to show that $w = f(z)$ rotates all straight lines (hence all tangents) passing through a point z_0 through the same angle $\alpha = \arg f'(z_0)$, but this follows from (3) by taking arguments. This in a nutshell is the proof, once the stage has been set.

Comment on Purpose of Section

Apart from applications, this discussion of geometric aspects of analytic functions should help the student gain a better understanding of complex functions. In a sense it is a counterpart of discussions of functions in terms of curves in calculus.

SOLUTIONS TO PROBLEM SET 12.5, page 678

2. $x = c$, $w = -y + ic$, horizontal lines; $y = k$, $w = -k + ix$, vertical lines

4. Only in size

6. The lower half-plane $v < 0$

8. From the last formula in Example 1 with $k = 1$ we have for $y = k = 1$ the image

$$v^2 = 4(1 + u) \qquad \text{or} \qquad u = \tfrac{1}{4}v^2 + 1,$$

a parabola opening to the right. For the boundary $y = 0$ we get $v^2 = 0$. The x-axis is "folded up" at 0, where angles are doubled, and is mapped onto the nonnegative ray of the u-axis.

10. $w = \dfrac{1}{x + iy} = \dfrac{x - iy}{x^2 + y^2}$. For $x = 1$ we thus have

$$u = \frac{1}{1 + y^2}, \qquad v = \frac{-y}{1 + y^2}.$$

Hence

$$u^2 + v^2 = \frac{1 + y^2}{(1 + y^2)^2} = \frac{1}{1 + y^2} = u.$$

This implies $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$. The image is the closed disk bounded by this circle.

12. $w' = 2(z^3 - a) \cdot 3z^2 = 0$ gives $z = 0$ and $z = \sqrt[3]{a}$.

14. $w' = -4z/(z^2 - 1)^2$ after simplification. Hence $z = 0$.

16. Ellipse $z(t) = 3 \cos t + i \sin t$. Advise students that other solutions are possible.

18. $z(t) = t + ikt^2$

20. **CAS PROJECT.** Orthogonality is a consequence of conformality because in the w-plane, $u = const$ and $v = const$ are orthogonal. (a) $u = x^4 - 6x^2y^2 + y^4$, $v = 4x^3y - 4xy^3$; (b) $u = x/(x^2 + y^2)$, $v = -y/(x^2 + y^2)$; (c) $u = (x^2 - y^2)/(x^2 + y^2)^2$, $v = -2xy/(x^2 + y^2)^2$; (d) $u = 2x/((1 - y)^2 + x^2)$, $v = (1 - x^2 - y^2)/((1 - y)^2 + x^2)$

SECTION 12.6. Exponential Function, page 679

Purpose. Sections 12.6–12.8 are devoted to the most important elementary functions in complex, which generalize the corresponding real functions, and we emphasize properties that are not apparent in real.

We also discuss the basic mapping properties of these functions. This is important for practical reasons (in connection with potential theoretic applications) as well as for creating a better understanding of the nature of these complex special functions. It is the analog of what we do all the time in calculus when we discuss real functions in terms of their graphs in the xy-plane.

Basic Properties of the Exponential Function

Derivative and functional relation as in real

Euler formula, polar form of z

Periodicity with $2\pi i$, fundamental region

$e^z \neq 0$ for all z

Conformality of the mapping $w = e^z$ for all z

SOLUTIONS TO PROBLEM SET 12.6, page 682

2. $e(\cos 1 + i \sin 1) \approx 1.47 + 2.29i$, e

4. $-0.0755 - 2.5846i$

6. $\sqrt{2}\, e^{\pi i/4}$

8. $\sqrt[n]{r} \exp(i(\theta + 2k\pi)/n)$, $k = 0, 1, \cdots, n - 1$; $r = |z|$

10. $e^{\pi i/4}, e^{-3\pi i/4}, e^{-\pi i/4}, e^{3\pi i/4}$

12. $z = \frac{1}{2}\ln 2 \pm n\pi i$, $n = 0, 1, \cdots$

14. $z = \ln 5 + (\arctan \frac{3}{4} \pm 2n\pi)i$

16. Annulus $1/e < |w| < e$ cut along the negative real axis

18. Whole w-plane except $w = 0$

20. TEAM PROJECT. (a) $e^{1/z}$ is analytic for all $z \neq 0$. $e^{\bar{z}}$ is not analytic for any z. The last function is analytic if and only if $k = 1$.

(b) (i) $e^x \sin y = 0$, $\sin y = 0$. *Answer:* On the horizontal lines $y = \pm n\pi$, $n = 0, 1, \cdots$. (ii) $e^{-x} < 1$, $x > 0$ (the right half-plane).
(iii) $e^{\bar{z}} = e^{x-iy} = e^x(\cos y - i \sin y) = \overline{e^x(\cos y + i \sin y)} = \overline{e^z}$. *Answer:* All z.

(d) $f' = u_x + iv_x = f = u + iv$, hence $u_x = u$, $v_x = v$. By integration,

$$u = c_1(y)e^x, \qquad v = c_2(y)e^x.$$

By the first Cauchy–Riemann equation,

$$u_x = v_y = c_2'e^x, \quad \text{thus} \quad c_1 = c_2' \quad (' = d/dy).$$

By the second Cauchy–Riemann equation,

$$u_y = c_1'e^x = -v_x = -c_2e^x, \quad \text{thus} \quad c_1' = -c_2.$$

Differentiating the last equation with respect to y, we get

$$c_1'' = -c_2' = -c_1, \quad \text{hence} \quad c_1 = a \cos y + b \sin y.$$

Now for $y = 0$ we must have

$$u(x, 0) = c_1(0)e^x = e^x, \quad c_1(0) = 1, \quad a = 1,$$
$$v(x, 0) = c_2(0)e^x = 0, \quad c_2(0) = 0.$$

Also, $b = c_1'(0) = -c_2(0) = 0$. Together $c_1(y) = \cos y$. From this,

$$c_2(y) = -c_1'(y) = \sin y.$$

This gives $f(z) = e^x(\cos y + i \sin y)$.

SECTION 12.7. Trigonometric Functions, Hyperbolic Functions, page 682

Purpose. Discussion of basic properties (including mapping properties) of trigonometric and hyperbolic functions, with emphasis on the relations between these two classes of functions as well as between them and the exponential function; here we see on an elementary level that investigation of special functions in complex can add substantially to their understanding.

SOLUTIONS TO PROBLEM SET 12.7, page 686

2. The right side is

$$\cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$
$$= \tfrac{1}{4}(e^{z_1} + e^{-z_1})(e^{z_2} + e^{-z_2}) + \tfrac{1}{4}(e^{z_1} - e^{-z_1})(e^{z_2} - e^{-z_2}).$$

If we multiply out, then because of the minus signs the products $e^{z_1}e^{-z_2}$ and $e^{-z_1}e^{z_2}$ cancel in pairs. There remains, as asserted,

$$2 \cdot \tfrac{1}{4}(e^{z_1+z_2} + e^{-z_1-z_2}) = \cosh(z_1 + z_2).$$

Similarly for the other formula.

4. $\cos 1 \cosh 1 - i \sin 1 \sinh 1 \approx 0.8337 - 0.9889i$

6. $i \sinh \pi \approx 11.5487i$ (same as Prob. 5)

8. $\cos 5 \sinh 4 + i \cosh 4 \sin 5 = 7.7411 - 26.1865i$

10. $\cosh z = 0$, $\cosh x \cos y = 0$, $\cos y = 0$, $y = \pm(2n + 1)\pi/2$, $\sinh x \sin y = 0$, $\sin y \neq 0$ for those y, hence $\sinh x = 0$, $x = 0$. *Answer:*

$$z = \pm(2n + 1)\frac{\pi i}{2}, \qquad n = 0, 1, \cdots.$$

12. $\sin x \cosh y = 1000$, $\cos x \sinh y = 0$, $x = \pi/2 \pm 2n\pi$, $\cosh y = 1000$, $\cosh y \approx e^y/2$ (y large), $e^y \approx 2000$, $y \approx 7.600\,902$ (which agrees with the 6D value of the solution of $\cosh y = 1000$). *Answer:* $z = \pi/2 \pm 2n\pi \pm 7.600\,902i$.

14. The region in the right half-plane bounded by the v-axis and the hyperbola $4u^2 - \frac{4}{3}v^2 = 1$ because for $x = 0$ formula (6b) reduces to

$$\sin iy = i \sinh y.$$

Thus $u = 0$ (the v-axis) is the left boundary of that region. For $x = \pi/6$ we obtain

$$\sin(\pi/6 + iy) = \sin(\pi/6)\cosh y + i\cos(\pi/6)\sinh y,$$

thus

$$u = \tfrac{1}{2}\cosh y, \qquad v = \tfrac{1}{2}\sqrt{3}\sinh y$$

and we obtain the right boundary curve of that region from

$$1 = \cosh^2 y - \sinh^2 y = 4u^2 - \tfrac{4}{3}v^2,$$

as asserted.

16. The region in the upper half-plane bounded by portions of the u-axis, the ellipse $u^2/\cosh^2 3 + v^2/\sinh^2 3 = 1$ and the hyperbola $u^2 - v^2 = \frac{1}{2}$.

Indeed, for $x = \pm\pi/4$ we get [see (6b)]

$$\sin(\pm\tfrac{1}{4}\pi + iy) = \sin(\pm\tfrac{1}{4}\pi)\cosh y + i\cos(\pm\tfrac{1}{4}\pi)\sinh y$$
$$= \pm(1/\sqrt{2})\cosh y + i(1/\sqrt{2})\sinh y$$

and from this

$$1 = \cosh^2 y - \sinh^2 y = 2u^2 - 2v^2, \qquad \text{thus} \qquad u^2 - v^2 = \tfrac{1}{2}.$$

For $y = 0$ we get $v = 0$ (the u-axis).

For $y = 3$ we get

$$u = \sin x \cosh 3, \qquad v = \cos x \sinh 3,$$

hence

$$1 = \sin^2 x + \cos^2 x = \frac{u^2}{\cosh^2 3} + \frac{v^2}{\sinh^2 3}.$$

18. The upper boundary maps onto the ellipse

$$\frac{u^2}{\cosh^2 1} + \frac{v^2}{\sinh^2 1} = 1$$

and the lower boundary onto the ellipse

$$\frac{u^2}{\cosh^2 \frac{1}{2}} + \frac{v^2}{\sinh^2 \frac{1}{2}} = 1.$$

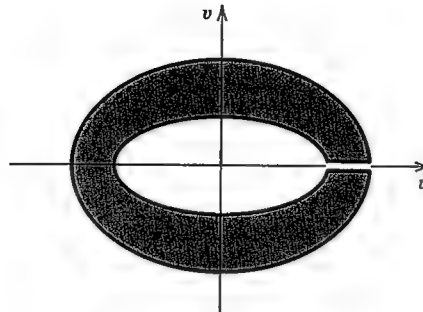
Since $0 < x < 2\pi$, we get the entire ellipses as boundaries of the image of the given domain, which therefore is an elliptical ring.

Now the vertical boundaries $x = 0$ and $x = 2\pi$ map onto the same segment

$$\sinh \tfrac{1}{2} \leqq u \leqq \sinh 1$$

of the u-axis because for $x = 0$ and $x = 2\pi$ we have

$$u = \cosh y, \qquad v = 0.$$

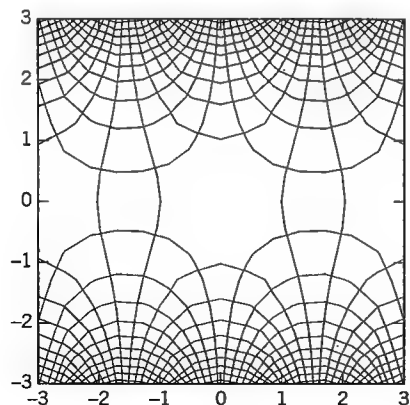
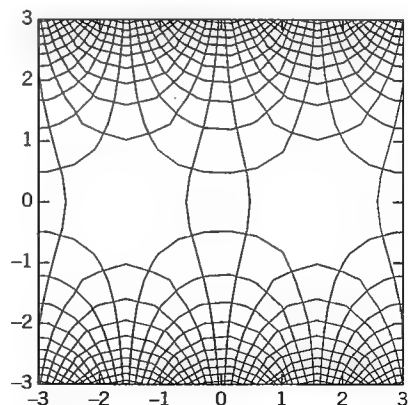
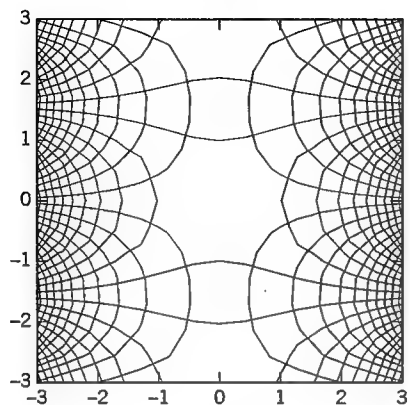
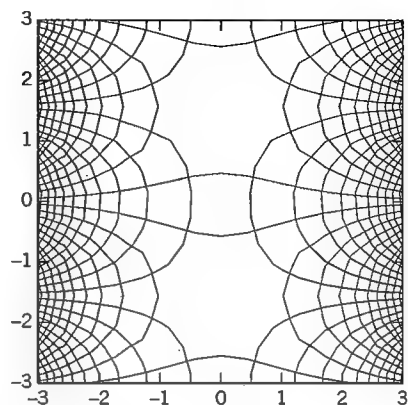
Answer: Elliptical annulus between those two ellipses and cut along that segment. See the figure.

Section 12.7. Problem 18

20. CAS PROJECT. This is an impressive demonstration of the relationships between the four functions. (*a*) and (*b*) reflect that they are translations of one another by an odd multiple of $\pi/2$. More about the actual formula $\cos z = \sin(z + \frac{1}{2}\pi)$ cannot be discovered from the plot. Similarly for (*c*) and (*d*), which are translates by multiples of $i\pi/2$ (thus in the y-direction). (*a*) and (*c*) are rotations of one another through 90°. Similarly for (*b*) and (*d*). Hence (*a*) and (*d*) are related by translations and rotations, and so are (*b*) and (*c*).

SECTION 12.8. Logarithm. General Power, page 687

Purpose. Discussion of the complex logarithm, which extends the real logarithm $\ln x$ (defined for positive x) to an infinitely many-valued relation (3) defined for all $z \neq 0$; definition of general powers z^c; mapping properties.

Comment on Notation

$\ln z$ is also denoted by $\log z$, but for the engineer, who also needs logarithms $\log x$ of base 10, the notation ln is more practical; this notation is widely used in mathematics.

Section 12.7. CAS Project 20

SOLUTIONS TO PROBLEM SET 12.8, page 691

4. $e^{\ln z} = e^{\ln|z| + i(\theta \pm 2n\pi)} = |z|e^{i\theta} = z,$
$\ln e^z = \ln|e^z| + i \arg e^z = \ln e^x + i(y \pm 2n\pi) = x + iy \pm 2n\pi i = z \pm 2n\pi i$

6. $\ln 20 - i \arctan(4/3) = 2.9957 - 2.2143i$

8. $\frac{1}{2}\ln 100.01 \pm (\pi - 0.0100)i = 2.302\,635 \pm 3.131\,593$

10. $1 \pm 2n\pi i,\ n = 0, 1, \cdots$

12. $\ln 4 \pm (2n + 1)\pi i,\ n = 0, 1, \cdots$

14. $\ln 5 + (\arctan(3/4) \pm 2n\pi)i,\ n = 0, 1, \cdots$

16. $e^{-2-3i/2} = e^{-2}(\cos\frac{3}{2} - i\sin\frac{3}{2}) = 0.010 - 0.135i$

18. $e^{e-\pi i} = -15.154$

20. $e^{2i\ln 2i} = e^{2i(\ln 2 + \pi i/2)} = e^{-\pi}(\cos(\ln 4) + i\sin(\ln 4)) = 0.0079 + 0.0425i$

22. $e^{(1-i)\ln(1+i)} = \exp[(1 - i)(\ln\sqrt{2} + \pi i/4)]$
$= \exp(\ln\sqrt{2} + \frac{1}{4}\pi i - i\ln\sqrt{2} + \frac{1}{4}\pi)$
$= \sqrt{2}\, e^{\pi/4}(\cos(\frac{1}{4}\pi - \ln\sqrt{2}) + i\sin(\frac{1}{4}\pi - \ln\sqrt{2}))$
$= 2.808 + 1.318i$

24. $e^{i\ln(1+3i)} = \exp[i(\ln\sqrt{10} + i\arctan 3)]$
$= e^{-\arctan 3}(\cos(\ln\sqrt{10}) + i\sin(\ln\sqrt{10}))$
$= 0.1168 + 0.2619i$

26. $e^{(2-4i)\ln(-1)} = e^{(2-4i)\pi i} = e^{4\pi} = 286751$

28. $e^{2\pi i \ln(1.3+0.4i)} = \exp[2\pi i(\ln\sqrt{1.85}) + i \arctan(0.4/1.3)]$

$= e^{-1.8755}(\cos(2\pi\ln\sqrt{1.85}) + i\sin(2\pi\ln\sqrt{1.85}))$

$= -0.0543 + 0.1433i$

30. TEAM PROJECT. (a) $w = \cos^{-1} z$, $z = \cos w = \frac{1}{2}(e^{iw} + e^{-iw})$. Multiply by $2e^{iw}$ to get a quadratic equation in e^{iw},

$$e^{2iw} - 2ze^{iw} + 1 = 0.$$

A solution is $e^{iw} = z + \sqrt{z^2 - 1}$, and by taking logarithms we get the given formula

$$\cos^{-1} z = w = -i\ln(z + \sqrt{z^2 - 1}).$$

(b) Similarly,

$$z = \sin w = \frac{1}{2i}(e^{iw} - e^{-iw}),$$

$$2ize^{iw} = e^{2iw} - 1,$$

$$e^{2iw} - 2ize^{iw} - 1 = 0,$$

$$e^{iw} = iz + \sqrt{-z^2 + 1}.$$

Now take logarithms, etc.

(c) $\cosh w = \frac{1}{2}(e^w + e^{-w}) = z$, $(e^w)^2 - 2ze^w + 1 = 0$, $e^w = z + \sqrt{z^2 - 1}$. Take logarithms.

(d) $z = \sinh w = \frac{1}{2}(e^w - e^{-w})$, $2ze^w = e^{2w} - 1$, $e^w = z + \sqrt{z^2 + 1}$. Take logarithms.

(e) $z = \tan w = \dfrac{\sin w}{\cos w} = -i\dfrac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = -i\dfrac{e^{2iw} - 1}{e^{2iw} + 1}$,

$$e^{2iw} = \frac{i - z}{i + z}, \qquad w = \frac{1}{2i}\ln\frac{i - z}{i + z} = \frac{i}{2}\ln\frac{i + z}{i - z}$$

(f) This is similar to (e).

SECTION 12.9. Linear Fractional Transformations. *Optional*, page 692

Purpose. Introduction to this large class of conformal mappings, also called **Möbius transformations.** These transformations form a group, have various general properties in common, and help to motivate the concept of the **extended complex plane,** which plays a more important role in advanced complex analysis than it does in our investigations.

Main Content, Important Concepts

Linear fractional transformations, special cases

Extended complex plane, point at infinity

Fixed points

Construction of linear fractional transformations

Mappings

Short Courses. This section may be omitted.

SOLUTIONS TO PROBLEM SET 12.9, page 698

2. The inverse is

$$z = \frac{w+i}{iw+1} = \frac{u+iv+i}{1-v+iu} = \frac{(u+iv+i)(1-v-iu)}{(1-v)^2+u^2}.$$

Multiplying out the numerator, a number of terms drop out, and the real part of the numerator is $2u$. This gives $\operatorname{Re} z = x$ in the form

$$x = c = \frac{2u}{(1-v)^2+u^2}.$$

This yields the circles

$$(v-1)^2 + \left(u - \frac{1}{c}\right)^2 = \frac{1}{c^2}$$

as claimed.

4. $z = (2w-1)/(-w+1)$. The equation for the fixed points of $w = f(z)$ is

$$z^2 + z - 1 = 0$$

with solutions

$$z = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

From the inverse we get the same equation with w instead of z. Of course, this is not surprising; a fixed point of the mapping must be a fixed point of the inverse.

6. TEAM PROJECT. (a) This follows by direct calculation and simplification.

(b) One can combine the cases of a straight line and a circle by writing ($A = 0$ a straight line, $A \neq 0$ a circle)

$$A(x^2+y^2) + Bx + Cy + D = 0 \qquad (A, B, C, D \text{ real}).$$

One can simplify the further work by writing this in terms of z and $\bar{z}$, a device that has other applications, too:

$$Az\bar{z} + B\frac{z+\bar{z}}{2} + C\frac{z-\bar{z}}{2i} + D = 0.$$

$w = 1/z$ gives $z = 1/w$. Substitution of this and multiplication by $w\bar{w}$ gives

$$A + B\frac{\bar{w}+w}{2} + C\frac{\bar{w}-w}{2i} + Dw\bar{w} = 0$$

or, in terms of u and v,

$$A + Bu - Cv + D(u^2+v^2) = 0,$$

which is a circle (if $D \neq 0$) or a straight line (if $D = 0$) in the w-plane.

(c) This follows by direct calculation.

(d) If we set

$$w_1 = cz, \qquad w_2 = w_1 + d, \qquad w_3 = \frac{1}{w_2}, \qquad w_4 = Kw_3,$$

then we have $w = w_4 + a/c$ from (c).

The statement to be proved is trivial for a translation or a rotation, fairly obvious for a uniform expansion or contraction, and true for an inversion, as proved in (b). Hence the statement is true for any LFT (1) because of (c).

(e) $w = -(z^2+i)/(iz^2+1)$

8. $w = z + 2$ **10.** $w = \dfrac{1}{z+1}$ **12.** $w = \dfrac{z}{(1-i)z + i}$ **14.** $w = iz$

16. The requirement is that

$$w = u = \frac{ax + b}{cx + d}$$

must come out real for all real x. Hence the four coefficients must be real, except possibly for a common complex factor.

18. $cz^2 - c = 0$ ($c \neq 0$) has those fixed points as solutions, and by comparing this with (5) we see that we must have

$$a - d = 0, \qquad b = c$$

and get the *answer*

$$w = \frac{az + c}{cz + a} \qquad (a, c \text{ arbitrary}).$$

SECTION 12.10. Riemann Surfaces. *Optional,* page 699

Purpose. To introduce the idea and some of the simplest examples of Riemann surfaces, on which multivalued relations become single-valued, that is, functions in the usual sense.

Short Courses. This section may be omitted.

SOLUTIONS TO PROBLEM SET 12.10, page 700

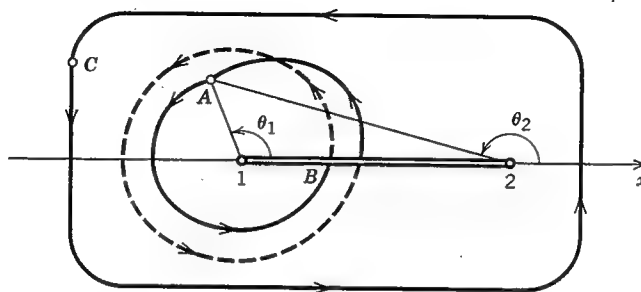
4. For $w = \sqrt[4]{z}$ the Riemann surface has 4 sheets. To them correspond in the w-plane the four angular regions of angle 90° each and bounded by the two bisecting lines of the four quadrants; $z = 0$ is a branch point. If z moves 4 times around the origin and back to its original position, then w completes a motion once around the origin.

Similarly for $w = \sqrt[5]{z}$, where we need 5 sheets and $z = 0$ is a branch point.

6. By the hint, we have

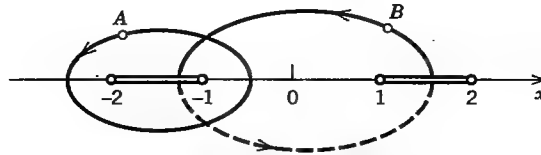
$$w = \sqrt{r_1}e^{i\theta_1/2}\sqrt{r_2}e^{i\theta_2/2} = \sqrt{r_1 r_2}e^{i(\theta_1+\theta_2)/2}.$$

If we move from A in the first sheet (see the figure), we get into the second sheet at B (dashed curve) and get back to A after two loops around the branch point 1.

Similarly for a loop around $z = 2$ (without encircling $z = 1$); this curve is not shown in the figure.

Section 12.10. Problem 6

If we move from C and back to C as shown, we do not cross the cut, we stay in the same sheet, and we increase θ_1 and θ_2 by 2π each. Hence $(\theta_1 + \theta_2)/2$ is increased by 2π, and we have completed one loop in the w-plane. This makes it plausible that two sheets will be sufficient for the present w and that the cut along which the two sheets are joined crosswise is properly chosen.

8. Branch points at ± 1 and ± 2, as shown in the figure, together with the cuts. If we pass a single cut, we get into the other sheet. If a path crosses two cuts, it is back in the sheet in which it started. The figure shows one path (A) that encircles two branch points and stays entirely in one sheet. The path from B and back to B also encloses two branch points, and since it crosses two cuts, part of it is in one sheet and part of it is in the other.

A discussion in terms of coordinates as in Prob. 6 would be similar to the previous one. Various other paths can be drawn and discussed in the figure.

Section 12.10. Problem 8

10. ± 1, $\pm i$; 2 sheets

SOLUTIONS TO CHAPTER 12 REVIEW, page 701

16. $-24 - 70i$

18. $\pm \sqrt[4]{74} \exp\left[\frac{i}{2} \text{Arc tan} \frac{-7}{5}\right] \approx 2.608 - 1.342i$

20. $1 + i$

22. $\sqrt{145} \exp\left[i \text{ Arc tan } \frac{1}{12}\right] = 12.042e^{0.0831i}$

24. $7.3e^{\pi i}$

26. $\pm 3, \pm 3i$

28. $(\pm 1 \pm i)/\sqrt{2}$

30. $\pm(4 - 4i)$

32. $1/z^2$

34. $\sin 2z$

36. $-22.72 + 49.65i$

38. $-2.3013i$

40. $|w| < 1/9, u > 0$

42. $0 < \text{Arg } w < 3\pi/2$

44. $e < |w| < e^2$ in the second quadrant

46. $z = 0, z^2 + 1 = 0, \pm\pi, \pm 2\pi, \cdots$

48. $w = (z + 2)/(z - i)$

50. $c^2z - (a - d)z - b = c(z + i)(z - i) = c(z^2 + 1)$ by the equation for the fixed points. By comparing powers of z we have $a - d = 0$, $b = -c$. Hence

$$w = \frac{az + b}{-bz + a}.$$

CHAPTER 13 Complex Integration

Change

We now discuss the two main integration methods (indefinite integration and integration by the use of the representation of the path) directly after the definition of the integral, postponing the proof of the first of these methods until Cauchy's integral formula is available in Sec. 13.2. This compactification of the material seems desirable from a practical point of view.

Comment

The introduction to the chapter mentions two reasons for the importance of complex integration. Another practical reason is the extensive use of complex integral representations in the higher theory of special functions; see Ref. [11] listed in Appendix 1.

SECTION 13.1. Line Integral in the Complex Plane, page 704

Purpose. To discuss the definition, existence, and general properties of complex line integrals. Complex integration is rich in methods, some of them very elegant. In this section we discuss the first two methods, integration by the use of path and (under suitable assumptions given in Theorem 1!) by indefinite integration.

Main Content, Important Concepts

Definition of the complex line integral

Existence

Basic properties

Indefinite integration (Theorem 1)

Integration by the use of path (Theorem 2)

Integral of $1/z$ around the unit circle (basic!)

ML-inequality (13) (needed often in our work)

Comment on Content

Indefinite integration will be justified in Sec. 13.2, after we have obtained Cauchy's integral theorem. We discuss this method here for two reasons: (i) to get going a little faster and, more importantly, (ii) to answer the students' natural question suggested by calculus, that is, whether the method works and under what condition—that it does not work unconditionally can be seen from Example 7!

SOLUTIONS TO PROBLEM SET 13.1, page 711

2. $4 + 3i - (9 + 4i)t \ (0 \leqq t \leqq 1)$

4. $3 \cos t + 2i \sin t \ (0 \leqq t \leqq 2\pi)$. Here (and elsewhere) one should emphasize the advantage of parametric representations, that one gets the entire curve, whereas $y = y(x)$ would give only the upper half (or the lower half), and $y'(x) \to \infty$ as $x \to \pm 3$.

6. $t + it^3 \ (-2 \leqq t \leqq 3)$

8. $2 \cosh t + i \sinh t \ (-\infty < t < \infty)$

10. Upper semicircle (radius $\sqrt{3}$, center $5i$)

12. $y = 3x^4 \ (-1 \leqq x \leqq 1)$

14. Hyperbola $xy = 4$ from $1 + 4i$ to $4 + i$

16. (1) $z(t) = 1 + it \ (1 \leqq t \leqq 2)$, $\dot{z}(t) = i$, $i \int_1^2 1 \, dt = i$.

(2) $z(t) = t + 2i \ (1 \leqq t \leqq 3)$, $\dot{z}(t) = 1$, $\int_1^3 t \, dt = 4$.

Answer: $4 + i$

18. $z(t) = t + it^2 \ (0 \leqq t \leqq 1)$, $\dot{z}(t) = 1 + 2it$, $\bar{z} = t - it^2$ gives

$$\int_0^1 (t - it^2)(1 + 2it) \, dt = 1 + \tfrac{1}{3}i.$$

20. $\text{Re}\, z^2 = x^2 - y^2$. (1) Upward, $z(t) = it$, $\dot{z}(t) = i$, $\int_0^1 - t^2 i \, dt = -\frac{1}{3}i$

(2) To the right, $z(t) = t + i$, $\dot{z}(t) = 1$, $\int_0^1 (t^2 - 1) \, dt = \frac{1}{3} - 1$

(3) Down, $z(t) = 1 + it$, t goes from 1 to 0, $\dot{z}(t) = i$, $\int_1^0 (1 - t^2) i \, dt = i(-1 + \frac{1}{3})$

(4) To the left, $z(t) = t$, t goes from 1 to 0, $\dot{z}(t) = 1$, $\int_1^0 t^2 \, dt = -\frac{1}{3}$.

Answer: $-1 - i$

22. By Theorem 1, the integral gives

$$\frac{\cosh \pi z}{\pi} \Bigg|_i^0 = \frac{1}{\pi}(1 - \cosh \pi i) = \frac{1}{\pi}(1 - \cos \pi) = \frac{2}{\pi}.$$

24. By Theorem 1 the integral gives

$$\tfrac{1}{4}e^{4z} \Bigg|_{8-3i}^{8-3i-\pi i} = \tfrac{1}{4}e^{32-12i}(e^{-4\pi i} - 1) = 0$$

because of (7) in Sec. 12.6.

26. $-3 \cdot 2\pi i$ by Example 6.

28. $L = \sqrt{5}$, $|\text{Re}\, z| = |x| \leqq 3 = M$, thus $3\sqrt{5} \geqq |4 + 2i| = \sqrt{20}$. It is typical that the bound is much larger than the actual value.

30. TEAM PROJECT. (b) (i) $12.8i$, (ii) $\frac{1}{2}(e^{2+4i} - 1)$

(c) The integral of $\text{Re}\, z$ equals $\frac{1}{2}\pi^2 - 2ai$. The integral of z equals $\frac{1}{2}\pi^2$. The integral of $\text{Re}\, z^2$ equals $\pi^3/3 - \pi a^2/2 - 2a\pi i$. The integral of z^2 equals $\pi^3/3$.

(d) The integrals of the four functions in (c) have for the present paths the values $\frac{1}{2}a\pi i$, 0, $(4a^2 - 2)i/3$, and $-2i/3$, respectively.

Parts (c) and (d) may also help to motivate our further discussions on path independence and the principle of deformation of path.

SECTION 13.2. Cauchy's Integral Theorem, page 713

Purpose. To discuss and prove the most important theorem in this chapter, Cauchy's integral theorem, which is basic by itself and has various basic consequences to be discussed in the remaining sections of the chapter.

Main Content, Important Concepts

Simply connected domain

Cauchy's integral theorem, Cauchy's proof

(Goursat's proof in Appendix 4)

Independence of path

Principle of deformation of path

Existence of indefinite integral

Extension of Cauchy's theorem to multiply connected domains

SOLUTIONS TO PROBLEM SET 13.2, page 720

2. (a) Yes. (b) No, since we would have to move the contour across $\pm 2i$ where $1/(z^2 + 4)$ is not analytic.

4. (a) $z = 0$ outside C, (b) $z = 0, \pm 1, \pm i$ outside C, (c) $0, \pm 3i$ outside C.

6. No, because of the principle of deformation of path.

8. 0, yes

10. $\dfrac{1}{\pi z - 1} = \dfrac{1/\pi}{z - 1/\pi}$. *Answer:* $\dfrac{1}{\pi} 2\pi i = 2i$ by the deformation principle and (6). No

12. $\displaystyle\int_0^{2\pi} e^{-3it} ie^{it}\, dt = -\tfrac{1}{2}e^{-2it}\Big|_0^{2\pi} = 0$, no

14. $\displaystyle\int_0^{2\pi} e^{it} ie^{it}\, dt = 0$, no

16. TEAM PROJECT. (b) (i) $\dfrac{2z + 3i}{z^2 + 1/4} = \dfrac{4}{z - i/2} - \dfrac{2}{z + i/2}$. From this, the principle of deformation of path, and (6) we obtain the *answer*

$$4 \cdot 2\pi i - 2 \cdot 2\pi i = 4\pi i.$$

(ii) Similarly,

$$\frac{z + 1}{z^2 + 2z} = \frac{1/2}{z} + \frac{1/2}{z + 2}.$$

Now $z = -2$ lies outside the unit circle. Hence the *answer* is $\frac{1}{2} \cdot 2\pi i = \pi i$.

(c) The integral of z, $\operatorname{Im} z$, z^2, $\operatorname{Re} z^2$, $\operatorname{Im} z^2$ equals $1/2$, $a/6$, $1/3$, $1/3 - a^2/30 - ia/6$, $a/6 - ia^2/30$, respectively. Note that the integral of $\operatorname{Re} z^2$ plus i times the integral of $\operatorname{Im} z^2$ must equal 1/3. Of course, the student should feel free to experiment with any functions whatsoever.

18. 0 by Cauchy's theorem because $z = 1$ and the portion $x > 1$ of the real axis lie outside the contour.

20. $\displaystyle\int_0^{\pi} x\, dx = 0$, $z(t) = e^{it}$, $0 \leqq t \leqq \pi$, hence the integral over the semicircle is

$$\int_0^{\pi} (\cos t) ie^{it}\, dt = i\int_0^{\pi} \tfrac{1}{2}(e^{it} + e^{-it})e^{it}\, dt = i[0 + \tfrac{1}{2}\pi] = \frac{\pi i}{2}.$$

22. $\dfrac{2z-1}{z(z-1)} = \dfrac{1}{z} + \dfrac{1}{z-1}$. *Answer:* $2\pi i + 2\pi i = 4\pi i$ by (6) because both 0 and 1 lie inside C.

24. 0 because the points $\pm 4n\pi i$, at which $\sinh z = 0$, lie all outside the contour of integration, so that Cauchy's integral theorem applies.

SECTION 13.3. Cauchy's Integral Formula, page 721

Purpose. To prove, discuss, and apply Cauchy's integral formula, the second major consequence of Cauchy's integral theorem (the first being the justification of indefinite integration).

Comment on Examples

The student has to find out how to write the integrand as a product $f(z)$ times $1/(z - z_0)$, and the examples (particularly Example 3) and problems are designed to give help in that technique.

SOLUTIONS TO PROBLEM SET 13.3, page 724

2. $2\pi i \cdot i/4 = -\pi/2$

4. The ellipse $\frac{1}{4}x^2 + y^2/\frac{1}{4} = 1$ includes the singularities at -1 and 1 in the interior, whereas $\pm i$ lie outside. If we write the integrand

$$\frac{z^2}{z^4-1} = \frac{\frac{1}{2}z^2}{(z^2+1)(z-1)} - \frac{\frac{1}{2}z^2}{(z^2+1)(z+1)}$$

we can apply Cauchy's integral formula to each of the two terms on the right and get $2\pi i \cdot \frac{1}{4} - 2\pi i \cdot \frac{1}{4} = 0$.

6. The integrand is

$$\frac{e^z}{\pi z - i} = \frac{e^z}{\pi} \cdot \frac{1}{z - i/\pi}.$$

By Cauchy's formula,

$$\frac{2\pi i}{\pi} e^{i/\pi} = 2i\left(\cos\frac{1}{\pi} + i\sin\frac{1}{\pi}\right) = -0.626 + 1.900i.$$

8. The integrand is

$$\frac{z^3 \sin z}{3z-1} = \frac{z^3 \sin z}{3} \cdot \frac{1}{z - 1/3}.$$

Hence Cauchy's formula gives

$$\frac{2\pi i}{81} \sin\frac{1}{3} = 0.02538i.$$

10. **TEAM PROJECT.** (a) Eq. (2) is

$$\oint_C \frac{z^3-6}{z-\frac{1}{2}i}\,dz = \left[\left(\frac{i}{2}\right)^3 - 6\right]\oint_C \frac{dz}{z-\frac{1}{2}i} + \oint_C \frac{z^3 - (\frac{1}{2}i)^3}{z-\frac{1}{2}i}\,dz$$

$$= \left(-\frac{i}{8} - 6\right)2\pi i + \oint_C \left(z^2 + \frac{1}{2}iz - \frac{1}{4}\right)dz = \frac{1}{4}\pi - 12\pi i$$

because the last integral is zero by Cauchy's integral theorem. The result agrees with that in Example 2, except for a factor 2.

(b) Using (12) in Appendix A3.1, we obtain (2) in the form

$$\oint_C \frac{\sin z}{z - \frac{1}{2}\pi}\,dz = \sin\tfrac{1}{2}\pi \oint_C \frac{dz}{z - \frac{1}{2}\pi} + \oint_C \frac{\sin z - \sin\frac{1}{2}\pi}{z - \frac{1}{2}\pi}\,dz$$

$$= 2\pi i + \oint_C \frac{2\sin\left(\frac{1}{2}z - \frac{1}{4}\pi\right)\cos\left(\frac{1}{2}z + \frac{1}{4}\pi\right)}{z - \frac{1}{2}\pi}\,dz.$$

As ρ in Fig. 343 approaches 0, the integrand approaches 0.

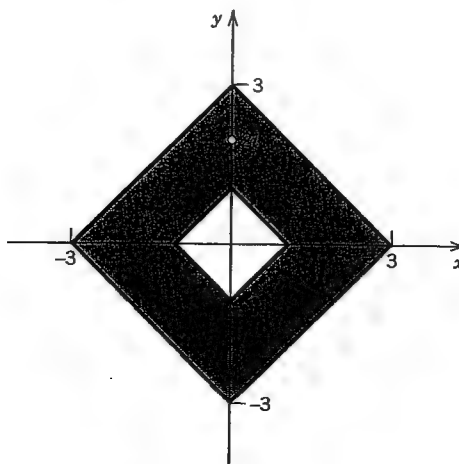
12. $z = 2$ lies outside the contour, so that we get the solution

$$\left.\frac{4 - \sin z}{z - 2}\right|_{z=0} = -4\pi i.$$

14. $z = 0$ lies inside the contour; the solutions of $e^z - 2i = 0$ lie outside because $e^z = 2i$, $z = \ln 2i \pm 2n\pi i$, and $|\ln 2i| > \ln 2 > \frac{1}{2}$. We thus obtain the *answer*

$$2\pi i \left.\frac{e^z}{e^z - 2i}\right|_{z=0} = \frac{2\pi i}{1 - 2i} = -0.8\pi + 0.4\pi i \approx -2.51 + 1.26i.$$

16. $4z^2 - 8iz = 4z(z - 2i) = 0$ at $z = 2i$ in the "ring" in the figure and at $z = 0$ not in the ring. Hence

$$\oint_C \frac{\sin z}{4z^2 - 8iz}\,dz = 2\pi i \cdot \frac{1}{4} \cdot \left.\frac{\sin z}{z}\right|_{z=2i} = \frac{\pi i}{2} \cdot \frac{\sin 2i}{2i} = \frac{\pi i}{4}\sinh 2 = 2.849i.$$

Section 13.3. Problem 16

18. $z_0 = i$ lies inside the large circle; $-i$ and -1 lie outside. The integral over $|z| = 0.2$ is zero by Cauchy's theorem. Hence

$$\oint_C \frac{\mathrm{Ln}\,(z + 1)}{(z - i)(z + i)}\,dz = 2\pi i \left[\frac{\mathrm{Ln}\,(z + 1)}{z + i}\right]_{z=i} = 2\pi i \frac{\mathrm{Ln}\,(1 + i)}{2i}$$

$$= \pi\left(\ln\sqrt{2} + \frac{\pi i}{4}\right) = 1.089 + 2.467i.$$

20. By Cauchy's integral theorem we can replace C by two small circles C_1 and C_2 around 1 and -1 and then apply (1) to get

$$\oint_{C_1} \left[\frac{\tan z}{z+1}\right] \frac{1}{z-1}\, dz + \oint_{C_2} \left[\frac{\tan z}{z-1}\right] \frac{1}{z+1}\, dz$$

$$= 2\pi i \left[\frac{\tan z}{z+1}\right]_{z=1} + 2\pi i \left[\frac{\tan z}{z-1}\right]_{z=-1} = 2\pi i \tan 1.$$

SECTION 13.4. Derivatives of Analytic Functions, page 725

Purpose. To discuss and apply the third major consequence of Cauchy's integral formula, the theorem on the existence and form of the derivatives of an analytic function.

Main Content

Formulas for the derivatives of an analytic function

Cauchy's inequality

Liouville's theorem

Morera's theorem (inverse of Cauchy's theorem)

Comments on Content

Technically the application of the formulas for derivatives in integration is practically the same as that in the last section.

The basic importance of (1) in giving the existence of all derivatives of an analytic function is emphasized in the text.

SOLUTIONS TO PROBLEM SET 13.4, page 729

2. $\displaystyle \frac{2\pi i}{1!} (e^{-z} \sin z)' \Big|_{z=0} = 2\pi i e^{-z}(-\sin z + \cos z)\Big|_{z=0} = 2\pi i$

4. $\displaystyle \frac{z^6}{(2z-1)^6} = \frac{z^6}{2^6(z-\frac{1}{2})^6}$. Hence the solution is

$$\frac{2\pi i}{5!} \cdot \frac{(z^2)^{(5)}}{2^6}\Big|_{z=1/2} = \frac{2\pi i}{120} \cdot \frac{720z}{2^6}\Big|_{z=1/2} = \frac{3\pi i}{32}.$$

6. Differentiating three times, we obtain the *answer*

$$\frac{2\pi i}{3!}(-\cos z)\Big|_{z=0} = -\frac{\pi i}{3}.$$

8. The answer is obtained by $2n$ differentiations, which reproduces $\cos z$ times a factor $(-1)^n$. Since $\cos 0 = 1$, we obtain $(-1)^n 2\pi i/(2n)!$.

10. From (1) we obtain

$$\frac{2\pi i}{2!} (z^2 e^z)''\Big|_{z=1/2} = \pi i(z^3 + 6z^2 + 6z)e^z\Big|_{z=1/2} = \frac{37\pi e^{1/2} i}{8}.$$

12. We have to differentiate twice, so that (1) gives

$$\frac{2\pi i}{2!} (z^3 + \sin z)''\Big|_{z=i} = \pi i(6z - \sin z)\Big|_{z=i} = (\sinh 1 - 6)\pi.$$

14. From (1) with $n = 1$ we obtain $2\pi i/(-\sin^2 z)\Big|_{z=\pi/2} = -2\pi i.$

16. From (1) we obtain

$$2\pi i(e^{z^2}/z)'\Big|_{z=2i} = 2\pi i e^{z^2}(2 - z^{-2})\Big|_{z=2i} = \tfrac{9}{2}\pi e^{-4} i.$$

18. $z = 2$ lies outside the contour, and (1) with $n = 1$ gives

$$2\pi i\left[\frac{\text{Ln}\,(z+3)}{z-2}\right]'_{z=-1} = 2\pi i\left(-\frac{1}{6} - \frac{\ln 2}{9}\right).$$

20. TEAM PROJECT. **(a)** If no such z existed, then $|f(z)| \leqq M$ for every $|z|$, which means that the entire function $f(z)$ would be bounded, hence a constant by Liouville's theorem.

(b) Let $f(z) = c_0 + c_1 z + \cdots + c_n z^n = z^n\left(c_n + \dfrac{c_{n-1}}{z} + \cdots + \dfrac{c_0}{z^n}\right)$, $c_n \neq 0$, $n > 0$. Set $|z| = r$. Then

$$|f(z)| > r^n\left(|c_n| - \frac{|c_{n-1}|}{r} - \cdots - \frac{|c_0|}{r^n}\right)$$

and $|f(z)| > \frac{1}{2}r^n|c_n|$ for sufficiently large r. From this the result follows.

(c) $|e^z| > M$ for real $z = x$ with $x > R = \ln M$. On the other hand, $|e^z| = 1$ for any pure imaginary $z = iy$ because $|e^{iy}| = 1$ for any real y (Sec. 12.6).

(d) If $f(z) \neq 0$ for all z, then $g = 1/f$ would be analytic for all z. Hence by (a) there would be values of z exterior to every circle $|z| = R$ at which, say, $|g(z)| > 1$ and thus $|f(z)| < 1$. This contradicts (b). Hence $f(z) \neq 0$ for all z cannot hold.

SOLUTIONS TO CHAPTER 13 REVIEW, page 730

16. $-64/35$

18. The four integrals along the four edges of the rectangle have the value 2, $-1 + \cosh 1$, $-2\cosh 1$, $-1 + \cosh 1$. The sum is 0.

20. $z = 0$ and $z = 2$ both lie inside the contour. Hence we obtain $-2\pi i - 2\pi i = -4\pi i$ (clockwise integration!).

22. $\dfrac{2\pi i}{3!}(e^z)'''\Big|_{z=1} = \dfrac{\pi i}{3}$

24. $z(t) = t + it^3$ $(0 \leqq t \leqq 3)$. Hence the integral takes the form

$$\int_0^3 t(1 + 3it^2)\,dt = \left[\frac{t^2}{2} + 3i\frac{t^4}{4}\right]_{t=0}^{3} = \frac{9}{2} + \frac{243}{4}i.$$

26. $2\pi i \dfrac{\pi}{\cos^2 \pi z}\Big|_{z=1} = 2\pi^2 i$

28. $z(t) = 3e^{it}$ $(0 \leqq t \leqq \frac{1}{2}\pi)$, $\dot{z} = 3ie^{it}$. Hence the integral is

$$\int_0^{\pi/2} \frac{1}{3}3ie^{it}\,dt = e^{\pi i/2} - 1 = i - 1.$$

30. $2\pi i \dfrac{\cos 4z}{4z^3}\Big|_{z=\pi/4} = -\dfrac{32i}{\pi^2}$

CHAPTER 14 Power Series Taylor Series

Major Change

Laurent series have been moved to Chap. 15, a better place because of their use in residue integration.

Furthermore, the section on uniform convergence (Sec. 14.5) has been made optional.

SECTION 14.1. Sequences, Series, Convergence Tests, page 732

Purpose. Since not too much changes in the transition from real to complex sequences and series, this section can almost be regarded as a review from calculus plus a presentation of convergence tests for later use.

Main Content, Important Concepts

Sequences, series, convergence, divergence

Comparison test (Theorem 5)

Ratio test (Theorem 8)

Root test (Theorem 10)

SOLUTIONS TO PROBLEM SET 14.1, page 740

2. Yes, no, $\pm(1 + i)$

4. Yes, no, ± 1, $\pm i$

6. No, because $z_n = (\cosh n\pi)/n$

8. Yes, yes, 0

10. Choose $\epsilon > 0$ arbitrary. By the definition of convergence there exists $N(\epsilon)$ such that $|z_n - l| < \frac{1}{2}\epsilon$, $|z_n{}^* - l^*| < \frac{1}{2}\epsilon$ for all $n > N(\epsilon)$. Hence for all these n,

$$|z_n + z_n{}^* - (l + l^*)| = |z_n - l + z_n{}^* - l^*| \leqq |z_n - l| + |z_n{}^* - l^*| < \epsilon.$$

This proves the assertion.

12. Convergent, the sum being e^{20+30i}.

14. Convergent since $\dfrac{1}{|n^2 + i|} < \dfrac{1}{n^2}$ and $\sum \dfrac{1}{n^2}$ converges.

16. This series converges by the ratio test because

$$\frac{((n + 1)!)^2}{(2n + 2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{(n + 1)^2}{(2n + 2)(2n + 1)} \quad \rightarrow \quad \frac{1}{4}.$$

18. Divergent since $1/\ln n > 1/n$ $(n = 2, 3, \cdots)$ and the harmonic series diverges.

20. TEAM PROJECT. (a) By the generalized triangle inequality (6), Sec. 12.2, we have

$$|z_{n+1} + \cdots + z_{n+p}| \leqq |z_{n+1}| + |z_{n+2}| + \cdots + |z_{n+p}|.$$

Since $|z_1| + |z_2| + \cdots$ converges by assumption, the sum on the right becomes less than any given $\epsilon > 0$ for every n greater than a sufficiently large N and $p = 1, 2, \cdots$, by Cauchy's convergence principle. Hence the same is true for the left side, which proves convergence of $z_1 + z_2 + \cdots$ by the same theorem.

(c) The form of the estimate of R_n suggests we use the fact that the ratio test is a comparison test based on the geometric series. This gives

$$R_n = w_{n+1} + w_{n+2} + \cdots = w_{n+1}\left(1 + \frac{w_{n+2}}{w_{n+1}} + \frac{w_{n+3}}{w_{n+1}} + \cdots\right),$$

$$\left|\frac{w_{n+2}}{w_{n+1}}\right| \leqq q, \quad \left|\frac{w_{n+3}}{w_{n+1}}\right| = \left|\frac{w_{n+3}}{w_{n+2}}\frac{w_{n+2}}{w_{n+1}}\right| \leqq q^2, \qquad \text{etc.},$$

$$|R_n| \leqq |w_{n+1}|\,(1 + q + q^2 + \cdots) = \frac{|w_{n+1}|}{1-q}.$$

(d) For this series we obtain the test ratio

$$\frac{1}{2}\left|\frac{n+1+i}{n+1}\cdot\frac{n}{n+i}\right| = \frac{n}{2(n+1)}\sqrt{\frac{(n+1)^2+1}{n^2+1}}$$

$$= \frac{1}{2}\sqrt{\frac{n^4+2n^3+2n^2}{n^4+2n^3+2n^2+2n+1}} < \frac{1}{2};$$

from this with $q = 1/2$ we have

$$|R_n| = \frac{|w_{n+1}|}{1-q} = \frac{|n+1+i|}{2^n(n+1)} = \frac{\sqrt{(n+1)^2+1}}{2^n(n+1)} < 0.05.$$

Hence $n = 5$ (by computation), and

$$s = \frac{31}{32} + \frac{661}{960}i = 0.96875 + 0.688542i.$$

Exact to 6 digits is $1 + 0.693147i$.

SECTION 14.2. Power Series, page 741

Purpose. To discuss the convergence behavior of power series, which is basic to our further work (and which is simpler than that of series having arbitrary complex functions as terms).

Proof of the Assertions in Example 6

$R = 1/\tilde{L}$ follows from $R = 1/\tilde{l}$ by noting that in the case of convergence, $\tilde{L} = \tilde{l}$ (the only limit point). $\tilde{l}$ exits by the Bolzano–Weierstrass theorem, assuming boundedness of $\{\sqrt[n]{|a_n|}\}$. Otherwise, $\sqrt[n]{|a_n|} > K$ for infinitely many n and any given K. Fix $z \neq z_0$ and take $K = 1/|z - z_0|$ to get

$$\sqrt[n]{|a_n(z-z_0)^n|} > K|z - z_0| = 1$$

and divergence for every $z \neq z_0$ by Theorem 9, Sec. 14.1.

Now, by the definition of a limit point, for a given $\epsilon > 0$ we have for infinitely many n

$$\tilde{l} - \epsilon < \sqrt[n]{|a_n|} < \tilde{l} + \epsilon,$$

hence for all $z \neq z_0$ and those n,

$$(*) \qquad (\tilde{l} - \epsilon)|z - z_0| < \sqrt[n]{|a_n(z-z_0)^n|} < (\tilde{l} + \epsilon)|z - z_0|.$$

The right inequality holds even for *all* $n > N$ (N sufficiently large), by the definition of a *greatest* limit point.

Let $\tilde{l} = 0$. Since $\sqrt[n]{|a_n|} \geqq 0$, we then have convergence to 0. Fix any $z = z_1 \neq z_0$. Then for $\epsilon = 1/(2|z_1 - z_0|) > 0$ there is an N such that $\sqrt[n]{|a_n|} < \epsilon$ for all $n > N$, hence

$$|a_n(z_1 - z_0)^n| < \epsilon^n |z_1 - z_0|^n = \frac{1}{2^n}$$

and convergence for all z_1 follows by the comparison test.

Let $\tilde{l} > 0$. We establish $1/\tilde{l}$ as the radius of convergence of (1) by proving

convergence of the series (1) if $|z - z_0| < 1/\tilde{l}$,

divergence of the series (1) if $|z - z_0| > 1/\tilde{l}$.

Let $|z - z_0| < 1/\tilde{l}$. Then, say, $|z - z_0|\tilde{l} = 1 - b < 1$. With this and $\epsilon = b/2|z - z_0| > 0$ in (*), for all $n > N$,

$$\sqrt[n]{|a_n(z - z_0)^n|} < \tilde{l}|z - z_0| + \epsilon|z - z_0| = 1 - b + \tfrac{1}{2}b < 1.$$

Convergence now follows from Theorem 9, Sec. 14.1.

Let $|z - z_0| > 1/\tilde{l}$. Then $|z - z_0|\tilde{l} = 1 + c > 1$. With this and $\epsilon = c/(2|z - z_0|) > 0$ in (*), for infinitely many n,

$$\sqrt[n]{|a_n(z - z_0)^n|} > \tilde{l}|z - z_0| - \epsilon|z - z_0| = 1 + c - \tfrac{1}{2}c > 1,$$

and divergence follows.

SOLUTIONS TO PROBLEM SET 14.2, page 745

2. $3, \infty$ **4.** $0, 4$ **6.** $\pi i, b/a$ **8.** $0, 1/\sqrt[3]{2}$

10. $-1, 4$ (the reciprocal of R in Example 5 of the text)

12. The quotient in (6) is

$$\frac{n^n}{n!}\Bigg/\frac{(n+1)^{n+1}}{(n+1)!} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1 + \dfrac{1}{n}\right)^n} \quad \rightarrow \quad \frac{1}{e}.$$

Hence the *answer* is $3i$, $1/e$.

14. $0, \infty$

16. $0, \sqrt{2}$ (not 2; see Team Project 20)

18. $0, 1/6$

20. TEAM PROJECT. (a) The faster the coefficients go to zero, the smaller $|a_{n+1}|$ is, compared to $|a_n|$, and the larger $|a_n/a_{n+1}| = R$ becomes.

(b) (i) Nothing. (ii) This multiplies R by $1/k$. (iii) The new series has radius of convergence $1/R$.

(c) In Example 6 we took the first term of one series, then the first term of the other, etc. We could have taken, for instance the first three terms of one series, then the first five terms of the other, then again three terms and five terms, etc. Or we could have mixed three or more series term by term.

(d) $\sum a_n z^{2n} = \sum a_n (z^2)^n$, $|z^2| < R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, hence $|z| < \sqrt{R}$

SECTION 14.3. Functions Given by Power Series, page 746

Purpose. To show what operations on power series are mathematically justified and to prove the basic fact that power series represent analytic functions.

Main Content

Termwise addition, subtraction, and multiplication of power series

Termwise differentiation and integration (Theorems 3, 4)

Analytic functions and derivatives (Theorem 5)

Comment on Content

That a power series is the Taylor series of its sum will be shown in the next section.

SOLUTIONS TO PROBLEM SET 14.3, page 750

2. Set $f = \sqrt[n]{n}$ and apply l'Hôpital's rule to $\ln f$,

$$\lim \ln f = \lim \frac{\ln n}{n} = \lim \frac{1/n}{1} = 0. \qquad \text{Hence} \qquad \lim f = 1.$$

4. 1

6. $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ consists of the fixed $k!$, which has no effect on R, and factors $n(n-1)\cdots(n-k+1)$, as obtained by differentiation. Since $\Sigma\,(z/\pi)^n$ has $R = \pi$, the answer is π.

8. $\sqrt{5/3}$. The root appears because of $z^{2n} = (z^2)^n$.

10. ∞. This is (36) in Appendix A3.1, except for a constant factor, and with z instead of x.

12. 1/4 because $1/(n+1)$ results from integration, and for the series without this factor in the coefficients we have in (6), Sec. 14.2,

$$\frac{(2n)!/(n!)^2}{(2n+2)!/((n+1)!)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} \quad \to \quad \frac{1}{4}.$$

14. ∞ because $3n(3n-1)$ results from differentiation, and for the coefficients without these factors we have in (6), Sec. 14.2,

$$\frac{1/n^n}{1/(n+1)^{n+1}} = \left(\frac{n+1}{n}\right)^n (n+1) \quad \to \quad \infty \qquad \text{as } n \to \infty.$$

16. This is a useful formula for binomial coefficients. It follows from

$$(1+z)^p(1+z)^q = \sum_{n=0}^{p} \binom{p}{n} z^n \sum_{m=0}^{q} \binom{q}{m} z^m$$

$$= (1+z)^{p+q} = \sum_{r=0}^{p+q} \binom{p+q}{r} z^r$$

by equating the coefficients of z^r on both sides. To get $z^n z^m = z^r$ on the left, we must have $n + m = r$; thus $m = r - n$, and this gives the formula in the problem.

18. The even-numbered coefficients are zero because $f(-z) = -f(z)$ implies

$$a_{2m}(-z)^{2m} = a_{2m}z^{2m} = -a_{2m}z^{2m}.$$

20. TEAM PROJECT. (a) The list is

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233.$$

In the recursion, a_n is the number of pairs of rabbits present and a_{n-1} is the number of pairs of offspring from the pairs of rabbits present at the end of the preceding month.

(b) Using the hint, we calculate

$$(1 - z - z^2)\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (a_n - a_{n-1} - a_{n-2})z^n = 1,$$

where $a_{-1} = a_{-2} = 0$, and Theorem 2 gives $a_0 = 1$, $a_1 - a_0 = 0$, $a_n - a_{n-1} - a_{n-2} = 0$ for $n = 2, 3, \cdots$. The converse follows from the uniqueness of a power series representation (see Theorem 2).

SECTION 14.4. Taylor Series and Maclaurin Series, page 751

Purpose. To derive and explain Taylor series, which include those for real functions known from calculus as special cases.

Main Content

Taylor series (1), integral formula (2) for the coefficients

Singularity, radius of convergence

Maclaurin series for e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh x$, $\text{Ln}(1 + z)$

Theorem 2 connecting Taylor series to the last section

Comment

The series just mentioned, with $z = x$, are familiar from calculus.

SOLUTIONS TO PROBLEM SET 14.4, page 757

2. $\sum_{n=0}^{\infty} z^{4n}$; $R = 1$, singularities at ± 1, $\pm i$

4. $1 - \frac{1}{2}z^2 + \frac{1}{8}z^4 - \frac{1}{48}z^6 + - \cdots$; $R = \infty$

6. $(2 - z)\sum_{n=1}^{\infty} nz^{n-1} = 2 + 3z + 4z^2 + 5z^3 + 6z^4 + \cdots$; $R = 1$

8. The series is

$$f = z + \frac{2}{1 \cdot 3}z^3 + \frac{2^2}{1 \cdot 3 \cdot 5}z^5 + \frac{2^3}{1 \cdot 3 \cdot 5 \cdot 7}z^7 + \cdots, \quad R = \infty.$$

It can be obtained in several ways. (a) Integrate the Maclaurin series of the integrand termwise and form the Cauchy product with the series of e^{z^2}. (b) f satisfies the differential equation $f' = 2zf + 1$. Use this, its derivatives $f'' = 2(f + zf')$, etc., $f(0) = 0$, $f'(0) = 1$, etc., and the coefficient formulas in (1). (c) Substitute $f = \sum_{n=0}^{\infty} a_n z^n$ and $f' = \sum_{n=0}^{\infty} n a_n z^{n-1}$ into the differential equation and compare coefficients; that is, apply the power series method (Sec. 4.1).

10. $z - \dfrac{z^3}{3!3} + \dfrac{z^5}{5!5} - \dfrac{z^7}{7!7} + - \cdots; \quad R = \infty$

12. $z - \dfrac{z^5}{2!5} + \dfrac{z^9}{4!9} - \dfrac{z^{13}}{6!13} + - \cdots; \quad R = \infty$

14. First of all, since $\sin(w + 2\pi) = \sin w$ and $\sin(\pi - w) = \sin w$, we obtain all values of $\sin w$ by letting w vary in a suitable vertical strip of width π, for example, in the strip $-\pi/2 \leqq u \leqq \pi/2$. Now since

$$\sin\left(\frac{\pi}{2} - iy\right) = \sin\left(\frac{\pi}{2} + iy\right) = \cosh y$$

and

$$\sin\left(-\frac{\pi}{2} - iy\right) = \sin\left(-\frac{\pi}{2} + iy\right) = -\cosh y,$$

we have to exclude a part of the boundary of that strip, so we exclude the boundary in the lower half-plane. To solve our problem we have to show that the value of the series lies in that strip. This follows from $|z| < 1$ and

$$\left|\mathrm{Re}\left(z + \frac{1}{2}\frac{z^3}{3} + \cdots\right)\right| \leqq \left|z + \frac{1}{2}\frac{z^3}{3} + \cdots\right| \leqq |z| + \frac{1}{2}\frac{|z|^3}{3} + \cdots$$

$$= \sin^{-1}|z| < \frac{\pi}{2}.$$

16. $\dfrac{1}{i(1 - i(z - i))} = -i\sum_{n=0}^{\infty} i^n (z - i)^n; \quad R = 1$

18. $[(z + 1) - 1]^5 = -1 + 5(z + 1) - 10(z + 1)^2 + 10(z + 1)^3 - 5(z + 1)^4 + (z + 1)^5$

20. $\cos \pi z = -\sin\left(\pi z - \dfrac{1}{2}\pi\right) = -\dfrac{\pi}{1!}\left(z - \dfrac{1}{2}\right) + \dfrac{\pi^3}{3!}\left(z - \dfrac{1}{2}\right)^3 - + \cdots; \quad R = \infty$

22. $1 + \dfrac{1}{2!}(z - \pi i)^2 + \dfrac{1}{4!}(z - \pi i)^4 + \dfrac{1}{6!}(z - \pi i)^6 + \cdots; \quad R = \infty$

24. $-\dfrac{1}{4} - \dfrac{2i}{8}(z - i) + \dfrac{3}{16}(z - i)^2 + \dfrac{4i}{32}(z - i)^3 - \dfrac{5}{64}(z - i)^4 + \cdots; \quad R = 2$

26. We obtain

$$\cos^2 z = \frac{1}{2} + \frac{1}{2}\cos 2z = \frac{1}{2} - \frac{1}{2}\cos(2z - \pi)$$

$$= \frac{1}{2}\left[\frac{4}{2!}\left(z - \frac{1}{2}\pi\right)^2 - \frac{4^2}{4!}\left(z - \frac{1}{2}\pi\right)^4 + - \cdots\right]; \quad R = \infty$$

28. TEAM PROJECT. **(a)** $(\text{Ln}(1 + z))' = 1 - z + z^2 - z^3 + - \cdots = \dfrac{1}{1 + z}$.

(c) $$\sin\left(z + \frac{1}{2}\pi\right) = \sum_{n=0}^{\infty} \frac{\sin^{(n)}(\frac{1}{2}\pi)}{n!} z^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z$$

(d) For $y \neq 0$ the series

$$\frac{\sin iy}{iy} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} (iy)^{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} y^{2n}$$

has positive terms; hence its sum cannot be 0.

SECTION 14.5. Uniform Convergence. *Optional,* page 759

Purpose. To explain uniform convergence and its application to power series (Theorem 1). To explain the two main reasons for the importance of uniform convergence (Theorems 2 and 3).

SOLUTIONS TO PROBLEM SET 14.5, page 766

2. This Maclaurin series of cosh z converges for all z. Use Theorem 1.

4. $\left|\dfrac{z^n}{|z|^{2n} + 1}\right| \leqq \dfrac{1}{|z|^n} \leqq \dfrac{1}{2^n}$ and $\Sigma\, 2^{-n}$ converges. Use the Weierstrass M-test.

6. $|\tanh^n |z|| \leqq 1$, $1/n(n + 1) < 1/n^2$, and $\Sigma\, 1/n^2$ converges. Use the Weierstrass M-test.

8. $R = 5$; uniform convergence for $|z| \leqq 5 - \delta$, $\delta > 0$.

10. $R = \infty$; uniform convergence on any bounded set.

12. $|\tanh n^2| \leqq 1$. Convergence for $|z^2| < 1/6$. Uniform convergence for $|z| \leqq 1/\sqrt{6} - \delta$, $\delta > 0$.

14. TEAM PROJECT. (a) Convergence follows from the comparison test (Sec. 14.1). Let $R_n(z)$ and $R_n{}^*$ be the remainders of (1) and (5), respectively. Since (5) converges, for given $\epsilon > 0$ we can find an $N(\epsilon)$ such that $R_n{}^* < \epsilon$ for all $n > N(\epsilon)$. Since $|f_m(z)| \leqq M_m$ for all z in the region G, we also have $|R_n(z)| \leqq R_n{}^*$ and therefore $|R_n(z)| < \epsilon$ for all $n > N(\epsilon)$ and all z in the region G. This proves that the convergence of (1) in G is uniform.

(b) Since $f_0' + f_1' + \cdots$ converges uniformly, we may integrate term by term, and the resulting series has the sum $F(z)$, the integral of the sum of that series. Therefore, the latter sum must be $F'(z)$.

(c) The converse is not true.

(d) Noting that this is a geometric series in powers of $q = (1 + z^2)^{-1}$, we have $q = |1 + z^2|^{-1} < 1$, $1 < |1 + z^2|^2 = (1 + x^2 - y^2)^2 + 4x^2y^2$, the exterior of a lemniscate. The series converges also at $z = 0$.

(e) We obtain

$$x^2 \sum_{m=1}^{\infty} \frac{1}{(1 + x^2)^m} = x^2\left[-1 + \sum_{m=1}^{\infty} \frac{1}{(1 + x^2)^m}\right]$$

$$= -x^2 + \frac{x^2}{1 - \dfrac{1}{1 + x^2}} = -x^2 + 1 + x^2 = 1.$$

16. $|B_n| = \dfrac{2}{L}\left|\int_0^L f(x)\sin\dfrac{n\pi x}{L}\,dx\right| < \dfrac{2}{L}ML$, where M is such that $|f(x)| < M$ on the interval of integration. Thus $|B_n| < K\ (= 2M)$. Now when $t \geqq t_0 > 0$,

$$|u_n| = \left|B_n \sin\frac{n\pi x}{L}\, e^{-\lambda_n^2 t}\right| < Ke^{-\lambda_n^2 t_0}$$

because $\left|\sin\dfrac{n\pi x}{L}\right| \leqq 1$ and the exponential function decreases in a monotone fashion as t increases. From this,

$$\left|\frac{\partial u_n}{\partial t}\right| = |-\lambda_n^2 u_n| = \lambda_n^2|u_n| < \lambda_n^2 K e^{-\lambda_n^2 t_0} \quad \text{when} \quad t \geqq t_0.$$

Consider $\sum_{n=1}^{\infty} \lambda_n^2 K e^{-\lambda_n^2 t_0}$. Since $\lambda_n = \dfrac{cn\pi}{L}$, the test ratio is

$$\frac{\lambda_{n+1}^2 K \exp(-\lambda_{n+1}^2 t_0)}{\lambda_n^2 K \exp(-\lambda_n^2 t_0)} = \left(\frac{n+1}{n}\right)^2 \exp\left[-(2n+1)\left(\frac{c\pi}{L}\right)^2 t_0\right] \to 0$$

as $n \to \infty$, and the series converges. From this and the Weierstrass test it follows that $\sum \dfrac{\partial u_n}{\partial t}$ converges uniformly and, by Theorem 4, has the sum $\dfrac{\partial u}{\partial t}$, etc.

SOLUTIONS TO CHAPTER 14 REVIEW, page 767

16. $z/(1-z)^2, \quad R = 1$

18. $R = \infty$

20. $R = 4$

22. $e^{-z^2}, \quad R = \infty$

24. $\left(1 - \dfrac{z-2i}{4+3i}\right)^{-1}, \quad R = 5$

26. $e^{2z} = e^{2(z-\frac{1}{2}\pi i)+\pi i} = -e^{2(z-\frac{1}{2}\pi i)} = -\sum_{n=0}^{\infty} \dfrac{2^n}{n!}(z - \tfrac{1}{2}\pi i)^n; \quad R = \infty$

28.

$$\frac{1}{(z+3-4i)^2} = \frac{1}{(3-4i)^2\left[1 + \dfrac{z}{3-4i}\right]^2}$$

$$= \frac{-1}{3-4i}\frac{d}{dz}\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{(3-4i)^n}$$

$$= \frac{(3+4i)^2}{25^2}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{3+4i}{25}\right)^n z^n$$

and

$$\left|\left(\frac{3+4i}{25}\right)z\right| < 1, \qquad |z| < \frac{25}{5} = 5, \qquad R = 5,$$

the distance of $3 + 4i$ from $z = 0$.

30. $\ln 3 + \frac{1}{3}(z-3) - \frac{1}{18}(z-3)^2 + \frac{1}{81}(z-3)^3 - \cdots; \quad R = 3$

32. $1 - 4i(z - i) - 6(z - i)^2 + 4i(z - i)^3 + (z - i)^4$, a binomial expansion also readily obtainable from Taylor's theorem.

34. $-\left(z - \frac{1}{2}\pi\right) + \frac{1}{3!}\left(z - \frac{1}{2}\pi\right)^3 - \frac{1}{5!}\left(z - \frac{1}{2}\pi\right)^5 + - \cdots$

$$= -\sin\left(z - \frac{1}{2}\pi\right); \quad R = \infty$$

36. $|30 + 10i| = \sqrt{1000} > |31 - 6i| = \sqrt{997}$. No, by Theorem 1 in Sec. 14.2. Smaller numbers serving the same purpose are $5 + 4i$ and $6 + 2i$; then $|5 + 4i| = \sqrt{41} > |6 + 2i| = \sqrt{40}$.

38. $|z| \leqq 7 - \delta, \delta > 0$

40. $R = 0$, the series converges only at the center $z = -1$, so that uniformity of convergence loses its meaning.

CHAPTER 15 Laurent Series, Residue Integration

Major Change

Laurent series, formerly in the previous chapter, have now been placed into this chapter, because of their main application, which is residue integration.

Applications to real integrals has been shortened because their practical importance seems to have decreased.

SECTION 15.1. Laurent Series, page 770

Purpose. Next in importance after power series are Laurent series, converging in an annulus, and we explain here their theory and technique of application.

Comment on Content

The Laurent series of a given function in a given annulus is unique; this is essential in view of our various methods and tricks of derivation. Because of our later work (residue integration!), two facts should be emphasized: (i) a function may have different Laurent series in different annuli with the same center, and (ii) the series converging in an immediate neighborhood of a singularity (except at the singularity itself) is of particular interest because it will give the residue (defined as the coefficient of the term of the power $1/z$).

SOLUTIONS TO PROBLEM SET 15.1, page 775

2. Divide the Maclaurin series of $\sin \pi z$ by z^2 to obtain the *answer*

$$\frac{\pi}{z} - \frac{\pi^3}{3!} z + \frac{\pi^5}{5!} z^3 - \frac{\pi^7}{7!} z^5 + - \cdots; \qquad 0 < |z| < \infty.$$

4. Using the sum formula for the geometric series, we obtain the *answer*

$$-\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots; \qquad 0 < |z| < 1.$$

6. $z^{-5} - 5z^{-4} + \frac{25}{2}z^{-3} - \frac{125}{6}z^{-2} + - \cdots; \qquad 0 < |z| < \infty$

8. Using the sum formula for the geometric series, we obtain

$$\frac{1}{z} e^z \frac{1}{1-z} = \frac{1}{z} \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{n=0}^{\infty} z^n = \frac{1}{z} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{1}{m!} \right) z^n$$

$$= \frac{1}{z} + 2 + \frac{5}{2} z + \frac{16}{6} z^2 + \frac{65}{24} z^3 + \cdots; \qquad 0 < |z| < 1.$$

10. The function is the same as in Prob. 8, but we now have the center $z_0 = 1$. (In Prob. 8 the center was 0.) We obtain

$$\frac{-e^{(z-1)+1}}{(z-1)(1+(z-1))} = \frac{-e}{z-1} e^{z-1} \sum_{m=0}^{\infty} (-1)^m (z-1)^m$$

$$= \frac{-e}{z-1} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{(-1)^m}{m!} \right) (z-1)^n$$

$$= e\left(-\frac{1}{z-1} - \frac{1}{2}(z-1) + \frac{1}{3}(z-1)^2 - \frac{3}{8}(z-1)^3 + - \cdots \right); \qquad R = 1.$$

12. $1 + \dfrac{2}{z-1} - \dfrac{3}{(z-1)^2}$

14. $\sin z = \sin(z - \frac{1}{4}\pi + \frac{1}{4}\pi) = \dfrac{1}{\sqrt{2}}(\sin(z - \frac{1}{4}\pi) + \cos(z - \frac{1}{4}\pi))$. This gives the *answer*

$$\frac{1}{\sqrt{2}}[(z - \tfrac{1}{4}\pi)^{-2} + (z - \tfrac{1}{4}\pi)^{-1} - \tfrac{1}{2} - \tfrac{1}{6}(z - \tfrac{1}{4}\pi) + \tfrac{1}{24}(z - \tfrac{1}{4}\pi)^2 + \cdots]; \quad R = \infty.$$

16. $e^{ab}[(z-b)^{-1} + a + \frac{1}{2}a^2(z-b) + \frac{1}{6}a^3(z-b)^2 + \cdots]; \quad R = \infty$

18. The *answer* is the Laurent series, which is the sum of the two series

$$\frac{1}{z-i} = \frac{1}{z(1 - i/z)} = \frac{1}{z} + \frac{i}{z^2} - \frac{1}{z^3} - \frac{i}{z^4} + \cdots \qquad (|z| > 1)$$

and

$$\frac{1}{z-2i} = \frac{-1}{2i(1 - z/2i)} = \tfrac{1}{2}i + \tfrac{1}{4}z - \tfrac{1}{8}iz^2 - \tfrac{1}{16}z^3 + \tfrac{1}{32}iz^4 + \cdots \qquad (|z| < 2).$$

20. The answer is the Taylor series that is the sum of the two series

$$\frac{1}{z-i} = \tfrac{1}{2}i + \tfrac{1}{4}(z+i) - \tfrac{1}{8}i(z+i)^2 - \tfrac{1}{16}(z+i)^3 + \tfrac{1}{32}i(z+i)^4 + \cdots$$

and

$$\frac{1}{z-2i} = \tfrac{1}{3}i + \tfrac{1}{9}(z+i) - \tfrac{1}{27}i(z+i)^2 - \tfrac{1}{81}(z+i)^3 + \tfrac{1}{243}i(z+i)^4 + \cdots.$$

22. TEAM PROJECT. (a) Let $\sum_{-\infty}^{\infty} a_n(z - z_0)^n$ and $\sum_{-\infty}^{\infty} c_n(z - z_0)^n$ be two Laurent series of the same function $f(z)$ in the same annulus. We multiply both series by $(z - z_0)^{-k-1}$ and integrate along a circle with center at z_0 in the interior of the annulus. Since the series converge uniformly, we may integrate term by term. This yields $2\pi i a_k = 2\pi i c_k$. Thus, $a_k = c_k$ for all $k = 0, \pm 1, \cdots$.

(b) No, because $\tan(1/z)$ is singular at $1/z = \pm\pi/2, \pm 3\pi/2, \cdots$, hence at $z = \pm 2/\pi$, $\pm 2/3\pi, \cdots$, which accumulate at 0.

(c) These series are obtained by termwise integration of the integrand. The second function is $\mathrm{Si}(z)/z^3$, where $\mathrm{Si}(z)$ is the sine integral [see (40) in Appendix A3.1]. *Answer:*

$$\frac{1}{z} + \frac{1}{2!2} + \frac{z}{3!3} + \frac{z^2}{4!4} + \cdots,$$

$$\frac{1}{z^2} - \frac{1}{3!3} + \frac{z^2}{5!5} - + \cdots.$$

SECTION 15.2. Singularities and Zeros. Infinity, page 776

Purpose. Singularities just appeared in connection with the convergence of Taylor and Laurent series in the last sections, and since we now have the instrument for their classification and discussion (i.e., Laurent series), this seems the right time for doing so. We also consider zeros, whose discussion is somewhat related.

Main Content, Important Concepts

Principal part of a Laurent series convergent near a singularity

Pole, behavior (Theorem 1)

Isolated essential singularity, behavior (Theorem 2)

Zeros are isolated (Theorem 3)

Relation between poles and zeros (Theorem 4)

Point ∞, extended complex plane, behavior at ∞

Riemann number sphere

SOLUTIONS TO PROBLEM SET 15.2, page 780

2. $\pm 1, \pm i$; fourth order

4. $(1 \pm 2n)\pi i/4$ (second order) because

$$\cosh 2z = \tfrac{1}{2}(e^{2z} + e^{-2z}) = 0, \quad e^{4z} = -1, \quad 4z = \ln(-1) = (1 \pm 2n)\pi i.$$

6. $\frac{1}{2}\pi \pm 2n\pi$, $n = 0, 1, \cdots$; tenth order

8. TEAM PROJECT. (a) $f(z) = (z - z_0)^n g(z)$ gives

$$f'(z) = n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z),$$

which implies the assertion because $g(z_0) \neq 0$.

(b) $f(z)$ as in (a) implies $1/f(z) = (z - z_0)^{-n} h(z)$, where $h(z) = 1/g(z)$ is analytic at z_0 because $g(z_0) \neq 0$.

(c) $f(z) - k = 0$ at those points. Apply Theorem 3.

(d) $f_1(z) - f_2(z)$ is analytic in D and zero at each z_n. Hence its zeros are not isolated because that sequence converges. Thus it must be constant since otherwise it would contradict Theorem 3. And that constant must be zero because it is zero at those points. Thus $f_1(z)$ and $f_2(z)$ are identical in D.

10. $\pm 1, \pm 3, \pm 5, \cdots$ (simple poles); ∞ (essential singularity)

12. ∞ (essential singularity)

14. $\pm 2i$ (essential singularities). These are the solutions of $z^2 + 4 = 0$. Also, $f(1/w) = \cosh [w^2/(1 + 4w^2)]$ is analytic at $w = 0$. Hence the given function is analytic at ∞.

16. $-i$ (essential singularity)

18. $\pi/4 \pm n\pi$ (simple poles). These are the points where the sine and cosine curves intersect. They have a different tangent there, hence their difference $\cos z - \sin z$ cannot have a zero derivative at those points; accordingly, those zeros are simple and give simple poles of the given function. To make sure that no further zeros of $\cos z - \sin z$ exist, one must calculate

$$\cos z - \sin z = \left(\frac{1}{2} - \frac{1}{2i}\right)e^{iz} + \left(\frac{1}{2} + \frac{1}{2i}\right)e^{-iz} = 0,$$

and by simplification,

$$e^{2iz} = i, \qquad z = \frac{\pi}{4} \pm n\pi, \qquad n = 0, 1, \cdots,$$

so that we get no further solutions beyond those found by inspecting those two curves.

20. For $|z|$ small enough we have $|1 + z| > 1/\sqrt{2}$, $|1 - z| > 1/\sqrt{2}$; hence $|1 - z^2| = |1 + z|\,|1 - z| > 1/2$ and

$$\left|\frac{1}{z^3} - \frac{1}{z}\right| = \frac{1}{|z|^3}\left|1 - z^2\right| > \frac{1}{2|z|^3} \quad \to \quad \infty \qquad \text{as} \qquad |z| \to 0.$$

This motivates the proof.

To prove the theorem, let $f(z)$ have a pole of mth order at some point $z = z_0$. Then

$$\begin{aligned} f(z) &= \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \cdots \\ &= \frac{b_m}{(z - z_0)^m}\left[1 + \frac{b_{m-1}}{b_m}(z - z_0) + \cdots\right], \qquad b_m \neq 0. \end{aligned}$$

For given $M > 0$, no matter how large, we can find a $\delta > 0$ so small that

$$\frac{|b_m|}{\delta^m} > 2M \qquad \text{and} \qquad \left|\left[1 + \frac{b_{m-1}}{b_m}(z - z_0) + \cdots\right]\right| > \frac{1}{2}$$

for all $|z - z_0| < \delta$. Then

$$|f(z)| > \frac{|b_m|}{\delta^m}\,\frac{1}{2} > M.$$

Hence $|f(z)| \to \infty$ as $z \to z_0$.

SECTION 15.3. Residue Integration Method, page 781

Purpose. To explain and apply this most elegant integration method.

Main Content, Important Concepts

Formulas for the residues at poles

Residue theorem (several singularities inside the contour)

Comment

The extension from the case of a single singularity to several singularities (residue theorem) is immediate.

SOLUTIONS TO PROBLEM SET 15.3, page 786

2. 0 (at 0)

4. -1 (at $(2n + 1)\pi/2$, $n = 0, \pm1, \pm2, \cdots$)

6. $(-1)^{n+1}$ (at $(2n + 1)\pi/2$, $n = 0, \pm1, \pm2, \cdots$)

8. $-\frac{16}{3}i$ (at $2i$), $\frac{1}{3}i$ (at $-i$)

10. $\left.\dfrac{(e^z)''}{2!}\right|_{z=\pi i} = -\dfrac{1}{2}$

12. $z^2 - 4z - 5 = (z + 1)(z - 5)$. Simple poles at -1 [residue $(-1 - 23)/(-1 - 5) = 4$ by (4)] and 5 [residue $(5 - 23)/(5 + 1) = -3$], both inside C. *Answer:* $2\pi i(4 - 3) = 2\pi i$.

14. Simple poles at $\pm\frac{1}{2}$. By (4) the residues are

$$\frac{z^2 \sin z}{8z} = \frac{1}{8} z \sin z.$$

This gives the *answer*

$$2\pi i[\tfrac{1}{8}\cdot\tfrac{1}{2}\sin\tfrac{1}{2} + \tfrac{1}{8}(-\tfrac{1}{2})\sin(-\tfrac{1}{2})] = \tfrac{1}{4}\pi i \sin\tfrac{1}{2}.$$

16. Simple poles at $-1, 0, 1$. Equation (4) gives $(e^z + z)/(3z^2 - 1)$, hence $(e^{-1} - 1)/2$, -1, $(e + 1)/2$, respectively. *Answer:* $2\pi i(\cosh 1 - 1)$.

18. Simple pole at $z = i/2$ with residue

$$\operatorname*{Res}_{z=i/2} \frac{\sinh z}{2z - i} = \frac{\sinh\frac{1}{2}i}{2} = \frac{i}{2}\sin\frac{1}{2}.$$

Answer: $-\pi \sin\frac{1}{2} = -1.506$.

20. $\sin 4z = 0$ at $0, \pm\pi/4$ (inside C), $\pm\pi/2, \cdots$ (outside C). This gives three simple poles at $-\pi/4, 0, \pi/4$ to be taken into account, with residues

$$\operatorname*{Res}_{z=z_0} \frac{e^{-z^2}}{\sin 4z} = \frac{e^{-z_0^2}}{4\cos 4z_0} = -\frac{e^{-\pi^2/16}}{4}, \frac{1}{4}, -\frac{e^{-\pi^2/16}}{4},$$

respectively, and by the residue theorem the *answer*

$$\frac{2\pi i}{4}(1 - 2e^{-\pi^2/16}) = -0.1245i.$$

SECTION 15.4. Evaluation of Real Integrals, page 787

Purpose 1. To show that certain classes of *real* integrals over finite or infinite intervals of integration can also be evaluated by residue integration.

Comment on Content

Since residue integration requires a closed path, one must have methods for producing such a path. We see that for the finite intervals in the text, this is done by (2), perhaps preceded by a translation and change of scale if another interval is given. (This is not shown in the text.) In the case of an infinite interval, we start from a finite one, close it by some curve in complex (here, a semicircle; Fig. 360), blow it up, and make assumptions on the integrand such that we can prove (once and for all) that the value of the integral over the complex curve added goes to zero.

Purpose 2. Extension of the second of the two methods just mentioned to integrals of practical interest in connection with Fourier integral representations (Sec. 10.8) and to discuss the case of singularities on the real axis.

SOLUTIONS TO PROBLEM SET 15.4, page 793

2. The denominator is

$$\begin{aligned}25 - 24\cos\theta &= 25 - 12(z + z^{-1}) = -12z^{-1}(z^2 - \tfrac{25}{12}z + 1)\\ &= -12z^{-1}(z - \tfrac{4}{3})(z - \tfrac{3}{4}).\end{aligned}$$

Two simple poles, at $z = 4/3$ (outside the contour) and at $z = 3/4$ (inside). From this and $d\theta = dz/iz$ we obtain the *answer*

$$2\pi i \cdot \frac{i}{12} \cdot \frac{1}{3/4 - 4/3} = \frac{2\pi}{7}.$$

4. Using (2), we obtain for the integral

$$\oint_C \frac{\frac{1}{2}(z + 1/z)}{[3 + (1/2i)(z - 1/z)]iz}\,dz = \oint_C \frac{z^2 + 1}{z[z^2 + 6iz - 1]}$$
$$= \oint_C \frac{z^2 + 1}{z(z - z_1)(z - z_2)}\,dz.$$

The residue at the simple pole $z = 0$ is $1/(-1) = -1$. The two other poles are at $z_1 = (-3 + \sqrt{8})i$ inside the unit circle and $z_2 = (-3 - \sqrt{8})i$ outside the unit circle. From (3), Sec. 15.3, we obtain at z_1 the residue

$$\frac{z_1^{\,2} + 1}{z_1(z_1 - z_2)} = \frac{-(-3 + \sqrt{8})^2 + 1}{(-3 + \sqrt{8})i \cdot 2\sqrt{8}i} = \frac{-16 + 6\sqrt{8}}{-16 + 6\sqrt{8}} = 1.$$

Answer: 0.

This also follows by noting that the integral from 0 to $\frac{1}{2}\pi$ equals minus the integral from $\frac{1}{2}\pi$ to π (set $\theta = \pi - \theta^*$) and the integral from π to $\frac{3}{2}\pi$ equals minus the integral from $\frac{3}{2}\pi$ to 2π.

6. Using (2), we obtain for the integral

$$\oint_C \frac{dz}{iz\left(5 - \dfrac{3}{2i}\left(z - \dfrac{1}{z}\right)\right)} = -\frac{2}{3}\oint_C \frac{dz}{z^2 - \dfrac{10}{3}iz - 1} = -\frac{2}{3}\oint_C \frac{dz}{(z - 3i)(z - i/3)}.$$

The residue at the pole at $i/3$ is

$$-\frac{2}{3} \cdot \frac{1}{i/3 - 3i} = -\frac{1}{4}i.$$

Answer: $2\pi i(-\frac{1}{4}i) = \frac{1}{2}\pi$.

8. The integral equals

$$\frac{1}{i}\oint_C \frac{1 + 2(z + z^{-1})}{z[17 - 4(z + z^{-1})]}\,dz = \frac{1}{-4i}\oint_C \frac{2z^2 + z + 2}{z(z - z_1)(z - z_2)}\,dz$$

where $z_1 = 1/4$ (inside the unit circle) and $z_2 = 4$ (outside) give simple poles. The residue at $z = 0$ is $2/z_1z_2 = 2$, and at $z = 1/4$ it is

$$\frac{2/16 + 1/4 + 2}{(1/4)(1/4 - 4)} = -\frac{38}{15}.$$

This gives the *answer*

$$\frac{2\pi i}{-4i}\left(2 - \frac{38}{15}\right) = \frac{4\pi}{15}.$$

10. Simple poles at $z_1 = (2 + 2i)/\sqrt{2}$ and $z_2 = (-2 + 2i)/\sqrt{2}$ in the upper half-plane (and at $(\pm 2 - 2i)/\sqrt{2}$ in the lower half-plane). From (4) in Sec. 15.3 we obtain the residues

$$1/4z_1^{\,3} = (-1 - i)/(32\sqrt{2}), \qquad 1/4z_2^{\,3} = (1 - i)/(32\sqrt{2}).$$

Answer: $\pi/(8\sqrt{2})$.

12. Third-order pole at $z = i$ (and at $z = -i$ in the lower half-plane) with residue

$$\frac{1}{2}\left[\frac{1}{(z+i)^3}\right]''_{z=i} = \frac{6}{(2i)^5}.$$

Answer: $2\pi i \cdot 6/(2i)^5 = 3\pi/8$.

14. Second-order pole at $z_1 = 1 + 2i$ in the upper half-plane (and at $z_2 = 1 - 2i$ in the lower) with residue

$$\left[\frac{1}{(z-1+2i)^2}\right]'_{z=z_1} = \frac{-2}{(z_1-z_2)^3} = \frac{-2}{(4i)^3} = \frac{1}{32i}.$$

Answer: $2\pi i(1/32i) = \pi/16$.

16. Simple poles at i and $3i$ in the upper half-plane (and at $-i$ and $-3i$ in the lower) with residues

$$\left.\frac{1}{(z+i)(z^2+9)}\right|_{z=i} = \frac{1}{16i}, \qquad \left.\frac{1}{(z^2+1)(z+3i)}\right|_{z=3i} = -\frac{1}{48i}.$$

Answer: $2\pi i(1/16i - 1/48i) = \pi/12$.

18. Simple pole at $z_1 = \frac{1}{2}(-1 + i\sqrt{3})$ (and at $\frac{1}{2}(-1 - i\sqrt{3}$ in the lower half-plane), with residue

$$\frac{e^{2iz_1}}{i\sqrt{3}} = \frac{e^{-\sqrt{3}}}{i\sqrt{3}}(\cos 1 - i \sin 1).$$

Answer: $-2\pi e^{-\sqrt{3}}(\sin 1)/\sqrt{3}$ [by (10)].

20. Second-order poles at $z_1 = i$ and $z_2 = -i$ (in the lower half-plane). By (5), Sec. 15.3, we get the residue

$$\left[\frac{e^{2iz}}{(z+i)^2}\right]'_{z=i} = \left.\frac{e^{2iz}}{(z+i)^3}[2i(z+i) - 2]\right|_{z=i} = \frac{e^{-2}}{(2i)^3}(-6) = -3e^{-2}i/4.$$

Multiplying the imaginary part $-3e^{-2}/4$ by -2π gives the *answer*

$$3\pi e^{-2}/2 = 0.6378.$$

22. $z^2 - 2iz = z(z - 2i)$ shows that we have simple poles at 0 and $2i$ with residues [by (3), Sec. 15.3]

$$\left.\frac{1}{z-2i}\right|_{z=0} = \frac{i}{2} \quad \text{and} \quad \left.\frac{1}{z}\right|_{z=2i} = -\frac{i}{2}.$$

The *answer* is

$$\pi i(i/2) + 2\pi i(-i/2) = \pi/2.$$

24. $z^3 - 1 = 0$ has the solutions $z_1 = 1$, $z_2 = -1$ on the real axis, $z_3 = i$ in the upper half-plane (and $-i$ in the lower). By (4), Sec. 15.3, the residues at the first three of these four simple poles are

$$\frac{1}{4}\cdot 1^3 = \frac{1}{4}, \qquad \frac{1}{4}(-1)^3 = -\frac{1}{4}, \qquad \frac{1}{4i^3} = \frac{i}{4},$$

so that (14) gives the *answer*

$$\pi i\left(\frac{1}{4} - \frac{1}{4}\right) + 2\pi i\left(\frac{i}{4}\right) = -\frac{\pi}{2}.$$

26. TEAM PROJECT. (b) The integral of e^{-z^2} along C is zero. Writing it as the sum of four integrals over the four segments of C we have

$$\int_{-a}^{a} e^{-x^2}\,dx + ie^{-a^2}\int_0^b e^{y^2-2ayi}\,dy + e^{b^2}\int_a^{-a} e^{-x^2-2ibx}\,dx + ie^{-a^2}\int_b^0 e^{y^2+2ayi}\,dy = 0.$$

Let $a \to \infty$. Then the terms having the factor e^{-a^2} approach zero. Taking the real part of the third integral, we thus obtain

$$-\int_{\infty}^{-\infty} e^{-x^2}\cos 2bx\,dx = 2\int_0^{\infty} e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_{-\infty}^{\infty} e^{-x^2}\,dx = e^{-b^2}\sqrt{\pi}.$$

Answer: $\frac{1}{2}e^{-b^2}\sqrt{\pi}$.

(c) Use the fact that the integrands are odd.

SOLUTIONS TO CHAPTER 15 REVIEW, page 794

22. $6\pi i$ because C contains only the pole at $z = 3$ in its interior.

24. $z^3 - 9z = z(z + 3)(z - 3) = 0$ at $z = -3, 0, 3$ gives simple poles, all three inside C: $|z| = 4$. From (4), Sec. 15.3, we get the residues

$$\operatorname*{Res}_{z=-3} \frac{15z + 9}{z^3 - 9z} = \frac{15z + 9}{3z^2 - 9}\bigg|_{z=-3} = \frac{-36}{18} = -2$$

and similarly, at 0 the value $9/(-9) = -1$ and at 3 the residue $54/18 = 3$. Since all three poles lie inside C, by the residue theorem we have to take the sum of all three residues, which is zero. *Answer:* 0.

26. Simple poles at $z = -1/2, 1/2$ with residues [by (4), Sec. 15.3]

$$\operatorname*{Res}_{z=z_0} \frac{z^2 \sin z}{4z^2 - 1} = \frac{z_0^2 \sin z_0}{8z_0} = \frac{1}{8} z_0 \sin z_0 = \frac{1}{16} \sin \frac{1}{2}.$$

Answer: $2\pi i \cdot 2 \cdot \frac{1}{16} \sin \frac{1}{2} = \frac{1}{4}\pi i \sin \frac{1}{2} = 0.3765i$.

28. $z^3 \exp z^4 = (\frac{1}{4} \exp z^4)'$, so that indefinite integration (because of independence of path) gives, with $(1 + i)^4 = -4$,

$$\frac{1}{4} \exp z^4 \bigg|_{1+i}^{1} = \frac{1}{4}(e - e^{-4}).$$

Answer: $(e - e^{-4})/4 = 0.6750$. No.

30. From the Maclaurin series of $\sin z$ we see that the residue is 0 for odd n and $(-1)^{(n+2)/2}/(n - 1)!$ for $n = 2, 4, \cdots$. Multiplication by $2\pi i$ gives the answer.

32. Simple pole at 0, residue [by (4), Sec. 15.3]

$$\frac{\cosh z}{(\sinh z)'} = 1.$$

Answer: $2\pi i$.

34. The integral equals

$$\oint_C \frac{(z - 1/z)/2i}{iz[3 + \frac{1}{2}(z + 1/z)]}\,dz = -\oint_C \frac{z^2 - 1}{z[z^2 + 6z + 1]}\,dz.$$

At the simple pole at $z = 0$ the residue is -1 (not counting the minus in front of the integral). At the simple pole at $-3 + \sqrt{8}$ (inside the unit circle) the residue is

$$\frac{(-3 + \sqrt{8})^2 - 1}{(-3 + \sqrt{8})2/8} = 1.$$

Answer: 0.

36. $\pi/60$

38. Simple pole at $z_1 = i/2$ in the upper half-plane (and at $-i/2$ in the lower) with residue $1/(8z_1) = -i/4$. *Answer:* $2\pi i(-i/4) = \pi/2$.

40. Poles at $z_1 = -\frac{1}{2} + i\sqrt{3}/2$ and $z_2 = -\frac{1}{2} - i\sqrt{3}/2$ (both simple). We need, using (3) in Sec. 15.3,

$$\operatorname*{Res}_{z=z_1} \frac{e^{iz}}{(z - z_1)(z - z_2)} = \frac{e^{iz_1}}{z_1 - z_2} = e^{-\sqrt{3}/2}(c - is)/(i\sqrt{3})$$

$$= -ie^{-\sqrt{3}/2}(c - is)/\sqrt{3},$$

where $c = \cos\frac{1}{2}$ and $s = \sin\frac{1}{2}$. Taking 2π times the real part of this, we get the *answer*

$$-2\pi e^{-\sqrt{3}/2}(\sin\tfrac{1}{2})/\sqrt{3} = -0.7315.$$

CHAPTER 16 Complex Analysis Applied to Potential Theory

This seems perhaps the most important justification for teaching complex analysis to engineers, and it also provides for nice applications of conformal mapping.

SECTION 16.1. Electrostatic Fields, page 799

Purpose. To show how complex analysis can be used to discuss and solve two-dimensional electrostatic problems and to demonstrate the usefulness of complex potential, a major concept in this chapter.

SOLUTIONS TO PROBLEM SET 16.1, page 802

2. $F(z) = 20z + 300$

4. $\Phi = 100(y + \frac{1}{2}x)$, $F(z) = 100(\frac{1}{2} - i)z$

6. $110(\ln r)/\ln 2 = 159 \ln r$ (with r measured in cm)

8. $\Phi(r) = 20(\ln r)/\ln 2 - 10$, $\Phi(3) = 21.70 > 20$

10. Yes, because near a source line its effect is much stronger than that of the other source line, and for a single source line, the equipotential lines are exactly concentric circles.

12. $\Phi = 110 - 50xy$

14. Compare the formulas for $\cos^{-1}$ and $\cosh^{-1}$ in Team Project 30, Sec. 12.8, and note that $v = const$ in $u + iv = \cos^{-1} z$ represents ellipses.

16. **CAS PROJECT.** (a) $x^2 - y^2 = c$, $xy = k$

(b) $xy = c$, $x^2 - y^2 = k$; the rotation caused by the multiplication by i leads to the interchange of the roles of the two families of curves.

(c) $x/(x^2 + y^2) = c$ gives $(x - 1/2c)^2 + y^2 = 1/4c^2$. Also, $-y/(x^2 + y^2) = k$ gives the circles $x^2 + (y + 1/2k)^2 = 1/4k^2$. All circles of both families pass through the origin.

(d) Another interchange of the families, compared to (c), $(y - 1/2c)^2 + x^2 = 1/4c^2$, $(x - 1/2k)^2 + y^2 = 1/4k^2$.

SECTION 16.2. Use of Conformal Mapping, page 804

Purpose. To show how conformal mapping helps in solving potential problems by mapping given domains onto simpler ones or onto domains for which the solution of the problem (subject to the transformed boundary conditions) is known.

SOLUTIONS TO PROBLEM SET 16.2, page 807

2. Figure 315, Sec. 12.6, shows D (a semi-infinite horizontal strip) and D^* (the upper half of the unit circular disk); and $\Phi = 2e^x \cos y\, e^x \sin y = e^{2x} \sin 2y = 0$ on $y = 0$ and $y = \pi$, and $\sin 2y$ on the vertical boundary $x = 0$ of D.

4. $\Phi = 2 \sin x \cos x \cosh y \sinh y = \frac{1}{2} \sin 2x \sinh 2y = 0$ if $x = 0$ or $x = \frac{1}{2}\pi$ or $y = 0$, and $\frac{1}{2} \sin 2x \sinh 2$ if $y = 1$.

8. TEAM PROJECT. We map $0 \mapsto r_0$, $2c \mapsto -r_0$, obtaining from (2) with $b = z_0$ the conditions

$$r_0 = -\frac{-z_0}{-1} = z_0, \qquad -r_0 = \frac{2c - z_0}{2z_0c - 1} = \frac{2c - r_0}{2r_0c - 1},$$

hence

$$r_0 = \frac{1}{2c}(1 - \sqrt{1 - 4c^2}).$$

r_0 is real for positive $c \leqq \frac{1}{2}$. Note that with increasing c the image (an annulus) becomes slimmer and slimmer.

10. $\Phi = 100(1 - (1/\pi) \operatorname{Arg}(z - 1))$, $F(z) = 100(1 + (i/\pi) \operatorname{Ln}(z - 1))$

12. $\pm i$ are fixed points, and straight lines are mapped onto circles (or straight lines). From this the assertion follows. (It also follows by setting $x = 0$ and calculating $|w|$.)

14. The function $z = Z^2$ maps the first quarter of $|Z| \leqq 1$ onto the upper half of the unit disk $|z| \leqq 1$, the segments $0 \leqq X \leqq 1$ and $0 \leqq Y \leqq 1$ being mapped into the x-axis, where the potential is zero (Fig. 372a). From this the result follows.

SECTION 16.3. Heat Problems, page 808

Purpose. To show that previous examples and new ones can be interpreted as potential problems in time-independent heat flow.

Comment on Interpretation Change

Boundary conditions of importance in one interpretation may be of no interest in another; this is about the only handicap in a change of interpretation.

SOLUTIONS TO PROBLEM SET 16.3, page 811

2. By inspection,

$$T(x, y) = 10 + 7.5(y - x),$$

the real part of

$$F(z) = 10 - 7.5(1 + i)z.$$

A systematic derivation is as follows. The boundary and boundary values suggest that $T(x, y)$ is linear in x and y,

$$T(x, y) = ax + by + c.$$

From the boundary conditions,

$$T(x, x - 4) = ax + b(x - 4) + c = -20, \qquad (1)$$

$$T(x, x + 4) = ax + b(x + 4) + c = 40. \qquad (2)$$

By addition,

$$2ax + 2bx + 2c = 20.$$

Since this is an identity in x, we must have $a = -b$ and $c = 10$. From this and (1),

$$-bx + bx - 4b + 10 = -20.$$

Hence $b = 7.5$. This agrees with our result obtained by inspection.

4. $T = 10 + 105(\operatorname{Arg} z)/\pi$

6. The lines of heat flow are perpendicular to the isotherms, and heat flows from higher to lower temperatures. Accordingly, heat flows from the portion of higher temperature of the unit circle $|Z| = 1$ to that kept at a lower temperature, along the circular arcs that intersect the isotherms at right angles.

Of course, as temperatures on the boundary we must choose values that are physically possible, for example, 20°C and 100°C.

8. TEAM PROJECT. (a) Arg z or Arg w is a basic building block when we have jumps in the boundary values. To get it as the real part of an analytic function (a logarithm), we have to multiply the logarithm by $-i$. Otherwise we just incorporate the real constants that appear in $T(x, y)$. *Answer:*

$$F^*(w) = T_1 - i\frac{T_2 - T_1}{\pi}\operatorname{Ln}(w - a),$$

$$T^*(u, v) = \operatorname{Re} F^*(w) = T_1 + \frac{T_2 - T_1}{\pi}\operatorname{Arg}(w - a).$$

(b) $T^* = \dfrac{T_0}{\pi}\operatorname{Arg}(w - 1) - \dfrac{T_0}{\pi}\operatorname{Arg}(w + 1)$. This is the real part of

$$F^*(w) = -i\frac{T_0}{\pi}[\operatorname{Ln}(w - 1) - \operatorname{Ln}(w + 1)].$$

On the u-axis both arguments are 0 for $u > 1$, one equals π if $-1 < u < 1$, and both equal π, giving $\pi - \pi = 0$ if $u < -1$.

(c) $w - a = z^2$. Hence $\operatorname{Arg}(w - a) = \operatorname{Arg} z^2 = 2 \operatorname{Arg} z$. Thus (a) gives

$$T_1 + \frac{2}{\pi}(T_2 - T_1)\operatorname{Arg} z$$

and we see that $T = T_1$ on the x-axis and $T = T_2$ on the y-axis are the boundary data.

Geometrically, the a in $w = a + z^2$ is a translation, and z^2 opens the quadrant up onto the upper half-plane, so that the result of (a) becomes applicable and gives the potential in the quadrant.

10. $(400/\pi)$ Arg z. This is quite similar to Example 3 because the smaller circular boundary is a line of heat flow, as it must be for an insulated part of the boundary.

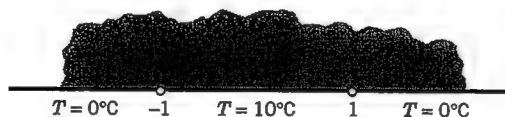
12. The answer is

$$\frac{10}{\pi}[\operatorname{Arg}(z^2 - 1) - \operatorname{Arg}(z^2 + 1)]$$

because $w = z^2$ maps the first quadrant onto the upper half-plane with $1 \mapsto 1$ and $i \mapsto -1$. The figure shows the transformed boundary conditions. The temperature is

$$\frac{10}{\pi}[\operatorname{Arg}(w - 1) - \operatorname{Arg}(w + 1)] = \frac{10}{\pi}[\operatorname{Arg}(z^2 - 1) - \operatorname{Arg}(z^2 + 1)],$$

in agreement with Team Project 8(b) with $T_0 = 10$.

14. $(200 \operatorname{Arg} z)/\pi$

Section 16.3. Problem 12

SECTION 16.4. Fluid Flow, page 812

Purpose. To give an introduction to complex analysis in potential problems of fluid flow.

Important Concepts

Stream function Ψ, streamlines $\Psi = const$

Velocity potential Φ, equipotential lines $\Phi = const$

Complex potential $F = \Phi + i\Psi$

Velocity $V = \overline{F'(z)}$

Circulation (6), vorticity, rotation (9)

Irrotational, incompressible

Flow around a cylinder (Example 2, Team Project 14)

SOLUTIONS TO PROBLEM SET 16.4, page 817

2. $F(z) = (1 - i)Kz/\sqrt{2}$, K positive real

4. $F(z) = iz^2 = i(x^2 - y^2) - 2xy$ gives the streamlines

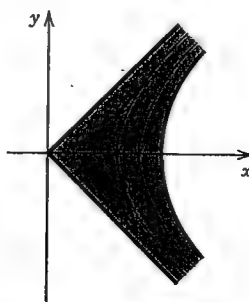
$$x^2 - y^2 = const.$$

The equipotential lines are

$$xy = const.$$

The velocity vector is

$$V = \overline{F'} = -2i\bar{z} = -2y - 2ix.$$

See the figure.

Section 16.4. Problem 4

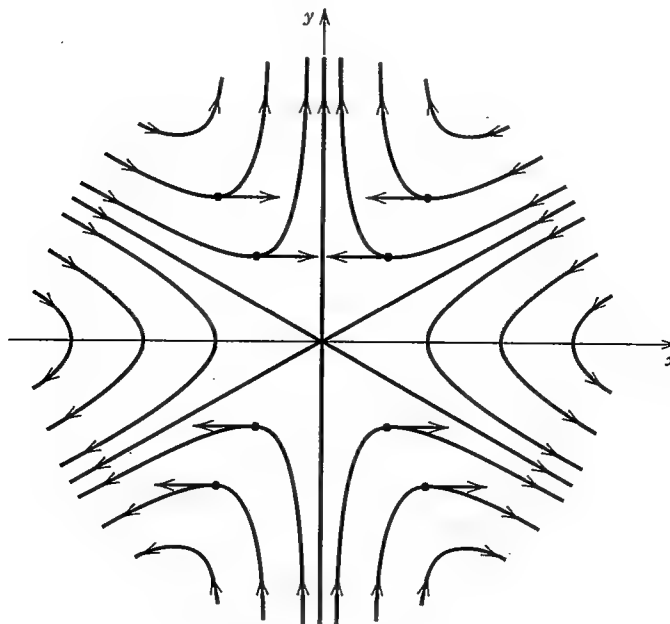
6. $F(z) = iz^3 = i(x^3 + 3ix^2y - 3xy^2 - iy^3) = -3x^2y + y^3 + i(x^3 - 3xy^2)$ gives the streamlines

$$x(x^2 - 3y^2) = const.$$

This includes the three straight-line asymptotes $x = 0$ and $y = \pm x/\sqrt{3}$ (which make 60° angles with one another, dividing the plane into six angular regions of angle 60° each), and we could interpret the flow as a flow in such a region. This is similar to the case $F(z) = z^2$, where we had four angular regions of 90° opening each (the four quadrants of the plane) and the streamlines were hyperbolas. In the present case the streamlines look similar but they are "squeezed" a little so that each stays within its region, whose two boundary lines it has for asymptotes.

The velocity vector is

$$V = -6xy + 3i(y^2 - x^2)$$

so that $V_2 = 0$ on $y = x$ and $y = -x$. See the figure.

Section 16.4. Problem 6

8. This rotates the whole flow pattern about the origin through the angle α.
10. $F(z) = z^2 + 1/z^2$, $\Psi = (r^2 - 1/r^2) \sin 2\theta = 0$ if $r = 1$ (the cylinder wall) or $\theta = 0$, $\pm\pi/2$, π. The unit circle and the axes are streamlines. For large $|z|$ the flow is similar to that in Example 1. For smaller $|z|$ it is a flow in the first quadrant around a quarter of $|z| = 1$. Similarly in the other quadrants.
12. $w = \cosh^{-1} z$ implies

$$z = x + iy = \cosh w = \cos iw = \sin (iw + \tfrac{1}{2}\pi).$$

Along with an interchange of the roles of the z- and w-planes, this reduces the present problem to the consideration of the sine function in Sec. 12.7 (compare with Fig. 316). Instead of (16), Sec. 12.7, we now have the hyperbolas

$$\frac{x^2}{\sin^2 c} - \frac{y^2}{\cos^2 c} = 1$$

where c is different from the zeros of sine and cosine, and as limiting cases, the y-axis and the two portions of the aperture.

14. **TEAM PROJECT.** (b) We have

$$F(z) = -\frac{iK}{2\pi} \ln z = -\frac{iK}{2\pi} \ln |z| + \frac{K}{2\pi} \arg z.$$

Hence the streamlines are circles

$$\frac{K}{2\pi} \ln |z| = const, \qquad \text{thus} \qquad |z| = const.$$

The formula also shows the asserted increase of the potential

$$\Phi(x, y) = \frac{K}{2\pi} \arg z$$

if $\arg z$ is increased by 2π.

(d) $F_1(z) = \frac{1}{2\pi} \ln (z + a)$ (source). $F_2(z) = -\frac{1}{2\pi} \ln (z - a)$ (sink). The minus sign has the consequence that the flow is directed radially inward toward the sink because the velocity vector V is

$$V = \overline{F'(z)} = -\frac{1}{2\pi} \cdot \frac{1}{\bar{z} - a} = -\frac{1}{2\pi} \cdot \frac{1}{x - iy - a}$$

$$= -\frac{1}{2\pi} \cdot \frac{x - a + iy}{(x - a)^2 + y^2} .$$

For instance, at $z = a + i$ (above the sink),

$$V = -\frac{i}{2\pi},$$

which is directed vertically downward, that is, in the direction of the sink at a.

(e) The addition gives

$$F(z) = z + \frac{1}{z} - \frac{iK}{2\pi} \ln z$$

$$= x + \frac{x}{x^2 + y^2} + \frac{K}{2\pi} \arg z + i\left(y - \frac{y}{x^2 + y^2} - \frac{K}{2\pi} \ln \sqrt{x^2 + y^2}\right) .$$

Hence the streamlines are

$$\Psi(x, y) = \operatorname{Im} F(z) = y - \frac{y}{x^2 + y^2} - \frac{K}{2\pi} \ln \sqrt{x^2 + y^2} = const.$$

In both flows that we have added, $|z| = 1$ is a streamline, hence the same is true for the flow obtained by the addition.

Depending on the magnitude of K, we may distinguish between three types of flow having either two or one or no stagnation points on the cylinder wall. The speed is

$$|V| = |\overline{F'(z)}| = |F'(z)| = \left|\left(1 - \frac{1}{z^2}\right) - \frac{iK}{2\pi z}\right| .$$

We first note that $|V| \to 1$ as $|z| \to \infty$; actually, $V \to 1$, that is, for points at a great distance from the cylinder the flow is nearly parallel and uniform. The stagnation points are the solutions of the equation $V = 0$, that is,

(A) $$z^2 - \frac{iK}{2\pi} z - 1 = 0.$$

We obtain

$$z = \frac{iK}{4\pi} \pm \sqrt{\frac{-K^2}{16\pi^2} + 1}.$$

If $K = 0$ (no circulation), then $z = \pm 1$, as in Example 2. As K increases from 0 to 4π, the stagnation points move from $z = \pm 1$ up on the unit circle until they unite at $z = i$. The value $K = 4\pi$ corresponds to a double root of equation (A). If $K > 4\pi$, the roots of (A) become imaginary, so that one of the stagnation points lies on the imaginary axis in the field of flow while the other one lies inside the cylinder, thus losing its physical meaning.

SECTION 16.5. Poisson's Integral Formula, page 819

Purpose. To represent the potential in a standard region (a disk $|z| \leqq R$) as an integral (5) over the boundary values; to derive from (5) a series (7) that gives the potential and for $|z| = R$ is the Fourier series of the boundary values.

Comment on Footnote 6

Poisson's discovery (1812) that Laplace's equation holds only outside the masses (or charges) resulted in the Poisson equation (Sec. 11.1). The publication on the Poisson distribution (Sec. 22.7) appeared in 1837.

SOLUTIONS TO PROBLEM SET 16.5, page 822

2. $\Phi = 2 - r\cos\theta$

4. $\Phi = r^4\cos 4\theta - r^2\cos 2\theta$

6. $\Phi = \frac{3}{8} + \frac{1}{2}r^2\cos 2\theta + \frac{1}{8}r^4\cos 4\theta$

8. $\Phi = \pi - 2\left(r\sin\theta + \dfrac{r^2}{2}\sin 2\theta + \dfrac{r^3}{3}\sin 3\theta + \cdots\right).$

Note that $\Phi(1, \theta)$ is neither even nor odd, but $\Phi(1, \theta) - \pi$ is odd, so that we get a sine series plus the constant term π.

10. $\Phi = \dfrac{1}{2} + \dfrac{2}{\pi}\left(r\sin\theta + \dfrac{1}{3}r^3\sin 3\theta + \dfrac{1}{5}r^5\sin 5\theta + \cdots\right)$

14. TEAM PROJECT. (a) $r = 0$ in (5) gives $\Phi(0) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\Phi(R, \alpha)\,d\alpha$. Note that the interval of integration has length 2π, not $2\pi R$.

(b) $\nabla^2 u = 0$, $u = g(r)h(\theta)$, $g''h + \dfrac{1}{r}g'h + \dfrac{1}{r^2}gh'' = 0$, hence by separating variables

$$r^2\frac{g''}{g} + r\frac{g'}{g} = n^2, \qquad \frac{h''}{h} = -n^2, \qquad h = a_n\cos n\theta + b_n\sin n\theta.$$

Also,

$$r^2g'' + rg' - n^2g = 0. \qquad \text{A solution is } r^n/R^n.$$

(c) By the Cauchy–Riemann equations,

$$\Psi_r = -\frac{1}{r}\Phi_\theta = -\sum_{n=1}^{\infty}\frac{r^{n-1}}{R^n}(-a_n\sin n\theta + b_n\cos n\theta)n,$$

$$\Psi = \Psi(0) + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n(-b_n\cos n\theta + a_n\sin n\theta).$$

(d) From the series for Φ and Ψ we obtain by addition

$$F(z) = a_0 + i\Psi(0) + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n [(a_n - ib_n)\cos n\theta + i(a_n - ib_n)\sin n\theta]$$

$$= a_0 + i\Psi(0) + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (a_n - ib_n)e^{in\theta},$$

$$a_n - ib_n = \frac{1}{\pi}\int_0^{2\pi} \Phi(R, \alpha)e^{-in\alpha}\, d\alpha.$$

Using $z = re^{i\theta}$, we have the power series

$$F(z) = a_0 + i\Psi(0) + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{R^n} z^n.$$

SECTION 16.6. General Properties of Harmonic Functions, page 822

Purpose. We derive general properties of analytic functions and from them corresponding properties of harmonic functions.

Main Content, Important Properties

Mean value of analytic functions over circles (Theorem 1)

Mean value of harmonic functions over circles, over disks (Theorem 2)

Maximum modulus theorem for analytic functions (Theorem 3)

Maximum principle for harmonic functions (Theorem 4)

Uniqueness theorem for the Dirichlet problem (Theorem 5)

Comment on Notation

Recall that we introduced F to reserve f for conformal mappings (beginning in Sec. 16.2), and we continue to use F also in this last section of Chap. 16.

SOLUTIONS TO PROBLEM SET 16.6, page 825

2. From (2) we obtain

$$F(0) = \frac{1}{2\pi}\int_0^{2\pi} 5r^4 e^{4i\alpha}\, d\alpha = 0,$$

as expected.

4. No, because $|z|$ is not analytic.

6. $z = 1 + e^{i\theta}$, $x = 1 + \cos\theta$, $y = \sin\theta$ gives

$$\frac{1}{2\pi}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta - \sin^2\theta)\, d\theta = \frac{1}{2\pi}(2\pi + 0 + \pi - \pi) = 1,$$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \Phi r\, dr\, d\theta = \frac{1}{\pi}\int_0^1 2\pi \cdot r\, dr = 1.$$

8. $x = 1 + \cos\theta$, $y = 1 + \sin\theta$, $\Phi = 3(1 + \cos\theta)^2(1 + \sin\theta) - (1 + \sin\theta)^3$. Integrate over θ from 0 to 2π, divide by 2π. This gives $2 = \Phi(1, 1)$.

10. TEAM PROJECT. (a) (i) Polar coordinates show that $|F(z)| = |z|^2$ assumes its maximum at the boundary point $4 + 7i$, namely, 65, but at no interior point. (ii) Use the fact that $|e^z| = e^x$ is monotone.

(b) $F(z)$ is not analytic.

(c) From (7), Sec. 12.7, we obtain in a small disk with center at $\pi/2$

$$\left|\sin\left(\frac{\pi}{2} + iy\right)\right|^2 = \sin^2\frac{\pi}{2} + \sinh^2 y$$
$$= 1 + \sinh^2 y > 1 \qquad (|y| > 0).$$

This shows that the maximum of $|\sin z|$ is taken on the boundary of the disk at $1 + ir$, r the radius of the disk, and equals

$$[1 + \sinh^2 r]^{1/2}.$$

(d) The extension is simple. Since the interior D of C is simply connected, Theorem 3 applies. The maximum of $|F(z)|$ is assumed on C, by Theorem 3, and if $F(z)$ had no zeros inside C, then, by Theorem 3, it would follow that $|F(z)|$ would also have its minimum on C, so that $F(z)$ would be constant, contrary to our assumption. This proves the assertion.

The fact that $|F(z)| = const$ implies $F(z) = const$ for any analytic function $F(z)$ was shown in Example 3, Sec. 12.4.

$F(z) = z, z^2, z^3, \cdots$ furnish examples.

12. $e^x \leqq e^b$, $e^x = e^b$ only at $x = b$, $\cos y \leqq 1$, $\cos y = 1$ only at $y = 0, 2\pi$, and $(b, 0)$ and $(b, 2\pi)$ lie on the boundary.

14. $\Phi = \exp(x^2 - y^2)\cos 2xy$, D: $|z| \leqq 1$, $x \geqq 0$, $y \geqq 0$. Yes, $(u_1, v_1) = (1, 0)$ is the image of $(x_1, y_1) = (1, 0)$; this is typical. (u_1, v_1) is found by noting that on the boundary (semicircle), $\Phi^* = e^u \cos(\sqrt{1 - u^2})$ increases monotone with u. Similarly for D.

SOLUTIONS TO CHAPTER 16 REVIEW, page 826

16. $\Phi = 20(1 - x + y)$, $F = 20 - 20(1 + i)z$

18. $\Phi = \dfrac{x + y}{x^2 + y^2} = \dfrac{1}{2c}$, $(x - c)^2 + (y - c)^2 = 2c^2$, circles through the origin with center on $y = x$.

20. $\Phi = \exp(x^2 - y^2)\sin 2xy$

22. Isotherms are the rays $\operatorname{Arg} z = const$. Heat flows along circular arcs from the higher to the lower temperature.

24. 43.22°C, which is obtained as follows. We have

$$T(r) = a \ln r + b$$

and at the outer cylinder,

$$(1) \qquad T(10) = a \ln 10 + b = 20$$

and from the condition to be achieved

$$(2) \qquad T(5) = a \ln 5 + b = 30.$$

(1) subtracted from (2) gives

$$a(\ln 5 - \ln 10) = 10, \qquad a = 10/\ln\tfrac{1}{2} = -14.43.$$

From this and (1)

$$b = 20 - a \ln 10 = 53.22.$$

Hence on the inner cylinder we should have

$$T(2) = a \ln 2 + b = 43.22.$$

26. $\Psi = x + y = const$, $V = 1 - i$, flow between parallel plates sloping downward (45°)

28. $V = \bar{z} + 1 = x + 1 - iy$

30. $F(z) = z/2 + 2/z$

32. $50 - \dfrac{200}{\pi}\left(r \cos \theta - \dfrac{1}{3} r^3 \cos 3\theta + \dfrac{1}{5} r^5 \cos 5\theta - + \cdots\right)$

34. $V = \overline{F'}(z) = 2\bar{z} - 2/\bar{z}^3 = 0$; solutions ± 1, $\pm i$

PART E. NUMERICAL METHODS

The subdivision into three chapters has been retained. All three chapters have been updated in the light of computer requirements and developments. A list of suppliers (with addresses etc.) has been included on p. 829 of the book.

CHAPTER 17 Numerical Methods in General

Major Changes

Updating of this chapter consists of the inclusion of ideas, such as error estimation by halfing, changes in Sec. 17.4 on splines, the presentation of adaptive integration and Romberg integration, and further error estimation techniques in integration.

SECTION 17.1. Introduction, page 831

Purpose. To familiarize the student with some facts of numerical work in general, regardless of the kind of problem or the choice of method.

Main Content, Important Concepts

Floating-point representation of numbers, overflow, underflow,

Rounding

Stability

Sources of errors

Error, relative error, error propagation

Short Courses. Mention the round-off rule and the definitions of error and relative error.

SOLUTIONS TO PROBLEM SET 17.1, page 836

2. -0.89217×10^2, 0.50000×10^6, -0.22137×10^{-2}

6. 29.9666, 0.0334; 29.9666, 0.0333705

8. -99.980, -0.020; -99.980, -0.020004

10. Use the last formula in (12), Appendix A3.1. Avoiding small differences of large numbers or expressions that may become nearly 0/0 is an important task in the design of algorithms. The problem illustrates that often a simple change in a formula may help.

12. -0.126×10^{-2}, -0.402×10^{-3}; -0.267×10^{-6}; -0.849×10^{-7}

14. $65.425 + 17.05905 = 82.48405 \leqq S \leqq 65.435 + 17.05915 = 82.49415$

16. $2 \cdot (9.5 \cdot 19.5 + 19.5 \cdot 29.5 + 29.5 \cdot 9.5) = 2081.5 \leqq A \leqq 2321.5$ [cm^2]

18. The proof is practically the same as that in the text. With the same notation we get

$$\begin{aligned} |\epsilon| &= |x + y - (\tilde{x} + \tilde{y})| \\ &= |(x - \tilde{x}) + (y - \tilde{y})| \\ &= |\epsilon_1 + \epsilon_2| \leqq |\epsilon_1| + |\epsilon_2| \leqq \beta_1 + \beta_2. \end{aligned}$$

20. Since $x_2 = 2/x_1$ and 2 is exact, $|\epsilon_r(x_2)| = |\epsilon_r(x_1)|$ by Theorem 1b. Since x_1 is rounded to 4S, we have $|\epsilon(x_1)| \leqq 0.005$, hence

$$|\epsilon_r(x_1)| \leqq 0.005/39.95.$$

This implies

$$\begin{aligned} |\epsilon(x_2)| = |\epsilon_r(x_2)x_2| &= |\epsilon_r(x_1)x_2| \\ &\leqq (0.005/39.95) \cdot 0.0506 \\ &< 0.00001. \end{aligned}$$

22. $61.2 - 7.5 \cdot 15.5 + 11.2 \cdot 3.94 + 2.80 = 61.2 - 116 + 44.1 + 2.80 = -7.9$

$$\begin{aligned} ((x - 7.5)x + 11.2)x + 2.8 &= (-3.56 \cdot 3.94 + 11.2)3.94 + 2.80 \\ &= (-14.0 + 11.2)3.94 + 2.8 = -11.0 + 2.80 \\ &= -8.2 \end{aligned}$$

Exact: -8.336016

SECTION 17.2. Solution of Equations by Iteration, page 838

Purpose. Discussion of the most important methods for solving equations $f(x) = 0$, a very important task in practice.

Main Content, Important Concepts

Solution of $f(x) = 0$ by iteration (3) $x_{n+1} = g(x_n)$

Condition sufficient for convergence (Theorem 1)

Newton (–Raphson) method (5)

Speed of convergence, order

Secant, bisection, false position methods

Comments on Content

Fixed-point iteration gives the opportunity to discuss the idea of a **fixed point,** which is also of basic significance in modern theoretical work (existence and uniqueness of solutions of differential, integral, and other functional equations).

The less important *method of bisection* and *method of false position* are included in the problem set.

SOLUTIONS TO PROBLEM SET 17.2, page 847

2. $x_0 = 1, x_1 = 0, x_2 = 1, x_3 = 0, \cdots$
$x_0 = 0.5, x_1 = 0.875, x_2 = 0.330, \cdots$
$x_0 = 2, x_1 = -7, x_2 = 344, x_3 = -40\,707\,583, \cdots$

4. $x = \frac{1}{5}(x^2 + 1.01 + 1.88/x)$; 1, 0.778, 0.806347, 0.798340, 0.800447, 0.799881, 0.800032, 0.799991, 0.800002, 0.799999, 0.800000 (exact)

6. $x = 1/\cosh x$; 1, 0.64805, 0.82140, 0.73706, $\cdots$ approaches 0.76501 (5S exact, 16 steps) in a nonmonotone fashion.

8. $x = x/(e^x \sin x)$; 0.5, 0.63256, 0.56838, $\cdots$ converges to 0.58853 (5S exact) in 14 steps.

10. CAS PROJECT. (a) This follows from the intermediate value theorem of calculus.
(b) Roots $r_1 = 1.56155$ (6S-value), $r_2 = -1$ (exact), $r_3 = -2.56155$ (6S-value). (1)

r_1, about 12 steps, (2) r_1, about 30 steps, (3) convergent to r_2, divergent, (4) convergent to 0, divergent, (5) r_3, about 7 steps, (6) r_2, divergent, (7) r_1, 4 steps; this is Newton.

12. $f(x) = x^k - c, \quad x_{n+1} = x_n - (x_n - c/x_n^{k-1})/k$

$$= \left(1 - \frac{1}{k}\right)x_n + \frac{c}{kx_n^{k-1}}.$$

In each case, x_4 is the first value that gives the desired accuracy, 1.414 214, 1.259 921, 1.189 207, 1.148 698.

14. 0.906180 (6S exact, 4 steps, $x_0 = 1$), also obtainable exactly by solving a quadratic equation in x^2.

16. 2, 2.452, 2.473; temperature 39.02°C

18. 21, 21.20870, 21.20575, 21.20575. A good x_0 is essential. $x_0 = 20$ would give a zero near 2.36, which has no meaning for Bessel functions since such an x is too small for the asymptotic formula considered.

20. $f(x) = f_1(x) - f_2(x) = 0$; 3, 2.498, 2.472, 2.473

22. 0.7, 0.577094, 0.534162, 0.531426, 0.531391

24. TEAM PROJECT. (a)

ALGORITHM REGULA FALSI (f, a_0, b_0, ϵ, N). Method of False Position

This algorithm computes an interval $[a_n, b_n]$ containing a solution of $f(x) = 0$ (f continuous) or a solution c_n.

INPUT: Initial interval $[a_0, b_0]$, tolerance ϵ, maximum number of iterations N.

OUTPUT: Interval $[a_n, b_n]$ containing a solution, or a solution c_n, or message of failure.

For $n = 0, 1, \cdots, N - 1$ do:

Compute $c_n = \dfrac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$.

If $f(c_n) = 0$ then OUTPUT c_n. Stop. [*Successful completion*]

Else continue.

If $f(a_n)f(c_n) < 0$ then set $a_{n+1} = a_n$ and $b_{n+1} = c_n$.

Else set $a_{n+1} = c_n$ and $b_{n+1} = b_n$.

If $b_{n+1} - a_{n+1} \leqq \epsilon$ then OUTPUT $[a_{n+1}, b_{n+1}]$. Stop

[*Successful completion*]

Else continue.

End

OUTPUT $[a_N, b_N]$ and message "Failure". Stop.

[*Unsuccessful completion; N iterations did not give an interval of length not exceeding the tolerance.*]

End REGULA FALSI

(b) 2.68910, **(c)** 0.64171, 1.55715

SECTION 17.3. Interpolation, page 848

Purpose. To discuss methods for interpolating (or extrapolating) given data (x_0, f_0), $\cdots$, (x_n, f_n), all x_j different, arbitrarily or equally spaced, by polynomials of degree not exceeding n.

Main Content, Important Concepts

Lagrange interpolation (4) (arbitrary spacing)

Error estimate (5)

Newton's divided difference formula (10) (arbitrary spacing)

Newton's difference formulas (14), (18) (equal spacing)

Short Courses. Lagrange's formula briefly, Newton's forward difference formula (14).

Comment on Content

For given data, the interpolation polynomial $p_n(x)$ is unique, regardless of the method by which it is derived. Hence the error estimate (5) is generally valid (provided f is $n + 1$ times continuously differentiable).

SOLUTIONS TO PROBLEM SET 17.3, page 860

2. This parallels Example 3. From (5) we get

$$\epsilon_1(9.3) = (x - 9)(x - 9.5)\left.\frac{f''(t)}{2}\right|_{x=9.3} = \frac{0.03}{t^2},$$

where $9 \leqq t \leqq 9.5$. Now the right side is a monotone function of t, hence its extrema occur at 9.0 and 9.5. We thus obtain

$$0.00033 \leqq a - \tilde{a} \leqq 0.00037.$$

This gives the *answer*

$$2.2300 \leqq a \leqq 2.2301.$$

2.2300 is exact to 4D.

4. From (5) we obtain

$$\epsilon_2(9.2) = (x - 9)(x - 9.5)(x - 11)\left.\frac{(\ln t)'''}{6}\right|_{x=9.2} = \frac{0.036}{t^3}.$$

The right side is monotone in t, hence its extreme values occur at the ends of the interval $9 \leqq t \leqq 11$. This gives

$$0.000\,027 \leqq \epsilon_2(9.2) = a - \tilde{a} \leqq 0.000\,050$$

and by adding $\tilde{a} = 2.2192$

$$2.2192 \leqq a \leqq 2.2193.$$

6. From

$$L_0(x) = x^2 - 20.5x + 104.5,$$

$$L_1(x) = \frac{1}{0.75}(-x^2 + 20x - 99),$$

$$L_2(x) = \frac{1}{3}(x^2 - 18.5x + 85.5)$$

and the 5S-values of the logarithm in the text we obtain

$$p_2(x) = -0.005\,233x^2 + 0.205\,017x + 0.775\,950.$$

This gives the values and errors

$$\begin{aligned}&2.2407, \quad \text{error } 0\\&2.3028, \quad \text{error } -0.0002\\&2.3517, \quad \text{error } -0.0003\\&2.4416, \quad \text{error } 0.0007\\&2.4826, \quad \text{error } 0.0023.\end{aligned}$$

It illustrates that in extrapolation one may usually get less accurate values. $p_2(x)$ would change if we took more accurate values of the logarithm.

8. $L_0 = -\frac{1}{6}(x-1)(x-2)(x-3)$, $L_1 = \frac{1}{2}x(x-2)(x-3)$, $L_2 = -\frac{1}{2}x(x-1)(x-3)$, $L_3 = \frac{1}{6}x(x-1)(x-2)$. From this and the data we obtain

$$p_3(x) = 1 + 0.039740x - 0.335187x^2 + 0.060645x^3$$

and $p_3(0.5) = 0.943654$ (6S exact 0.938470), $p_3(1.5) = 0.510116$ (6S exact 0.511828), $p_3(2.5) = -0.047993$ (6S exact -0.048384); see Ref. [1], p. 390, in Appendix 1.

10. From (5) we obtain

$$\epsilon_2(0.75) = (x-0.25)(x-0.5)(x-1)\left.\frac{f'''(t)}{6}\right|_{x=0.75} = -0.005208f'''(t)$$

where, by differentiation,

$$f'''(t) = \frac{-4}{\sqrt{\pi}}(1-2t^2)e^{-t^2}.$$

Another differentiation shows that f''' is monotone on the interval $0.25 \leqq t \leqq 1$ because

$$f^{iv} = -\frac{8t}{\sqrt{\pi}}(-3+2t^2)e^{-t^2} \neq 0$$

on that interval. Hence the extrema of f''' occur at the ends of the interval, so that we obtain

$$-0.00433 \leqq a - \tilde{a} \leqq 0.00967$$

and by adding $\tilde{a} = 0.70929$

$$0.70496 \leqq a \leqq 0.71896.$$

Exact: 0.71116 (5D).

12. The difference table is

x_j	$f(x_j)$	1st Diff.	2nd Diff.	3rd Diff.
1.0	0.94608			
		0.37860		
1.5	1.32468		−0.09787	
		0.28073		−0.01002
2.0	1.60541		−0.10789	
		0.17284		
2.5	1.77825			

The interpolating polynomials and errors are

$$P_1(1.25) = f(1.0) + 0.5 \cdot 0.37860 = 1.13538 \quad (\epsilon = 0.01107)$$

$$p_2(1.25) = p_1(1.25) + \frac{0.5(-0.5)}{2} \cdot (-0.09787) = 1.14761 \ (\epsilon = -0.00116)$$

$$p_3(1.25) = p_2(1.25) + \frac{0.5(-0.5)(-1.5)}{6} \cdot (-0.01002) = 1.14699 \ (\epsilon = -0.00054)$$

Note the decrease of the error.

14. The divided difference table is

x_j	$f(x_j)$	$f[x_j, x_{j+1}]$	$f[x_j, x_{j+1}, x_{j+2}]$
9.0	2.1972		
		0.1082	
9.5	2.2513		−0.0053
		0.0977	
11.0	2.3979		

This gives by (10)

$$\begin{aligned} p_2(x) &= 2.1972 + (x - 9.0) \cdot 0.1082 + (x - 9.0)(x - 9.5)(-0.0053) \\ &= -0.0053x^2 + 0.2062x + 0.7702, \end{aligned}$$

the discrepancies being due to round-off, as can be seen by using one or two additional digits in the computations.

16. With the change in j the difference table is

j	x_j	$f_j = \cosh x_j$	∇f_j	$\nabla^2 f_j$	$\nabla^3 f_j$
−3	0.5	1.127 626			
			0.057 839		
−2	0.6	1.185 465		0.011 865	
			0.069 704		(0.000 697)
−1	0.7	1.255 169		(0.012 562)	
			(0.082 266)		
0	0.8	(1.337 435)			

From this and (18) we obtain

$$\begin{aligned} p_3(x) = 1.337\,435 &+ 0.082\,266 \cdot \frac{x - 0.8}{0.1} \\ &+ 0.012\,562 \cdot \frac{(x - 0.8)(x - 0.7)}{0.01 \cdot 2!} \\ &+ 0.000\,697 \cdot \frac{(x - 0.8)(x - 0.7)(x - 0.6)}{0.001 \cdot 3!} \end{aligned}$$

and with $x = 0.56$ this becomes

$$\begin{aligned} &1.337\,435 + 0.082\,266(-2.4) + 0.012\,562(-2.4)(-1.4)/2 \\ &\qquad + 0.000\,697(-2.4)(-1.4)(-0.4)/6 = 1.160\,945. \end{aligned}$$

This agrees with Example 5. The correct last digit is 1 (instead of 5 here or 4 in Example 5).

18. The difference table is

x_j	$J_1(x_j)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0.0	0.00000					
		9950				
0.2	0.09950		−297			
		9653		−289		
0.4	0.19603		−586		22	
		9067		−267		5
0.6	0.28670		−853		27	
		8214		−240		
0.8	0.36884		−1093			
		7121				
1.0	0.44005					

From this and (14) we get by straightforward calculation

$$p_5(x) = 0.00130x^5 + 0.00312x^4 - 0.06526x^3 + 0.00100x^2 + 0.49988x.$$

This gives as the values of $J_1(x)$, $x = 0.1(0.2)0.9$,

$$0.04993, \quad 0.14832, \quad 0.24227, \quad 0.32899, \quad 0.40595,$$

the errors being 1, 0, 0, 1, 0 unit of the last given digit.

20. TEAM PROJECT. (a) For $p_1(x)$ we need

$$L_0 = \frac{x - x_1}{x_0 - x_1} = 19 - 2x, \qquad L_1 = \frac{x - x_0}{x_1 - x_0} = -18 + 2x$$

$p_1(x) = 2.19722(19 - 2x) + 2.25129(-18 + 2x) = 1.22396 + 0.10814x$,
$p_1(9.2) = 2.21885$.
Exact 2.21920, error 0.00035. For p_2 we need

$$L_0 = 104.5 - \tfrac{41}{2}x + x^2$$
$$L_1 = -132 + \tfrac{80}{3}x - \tfrac{4}{3}x^2$$
$$L_2 = 28.5 - \tfrac{37}{6}x + \tfrac{1}{3}x^2$$

This gives (with 10S- values for the logarithm)

$$p_2(x) = 0.779466 + 0.204323x - 0.0051994x^2,$$

hence $p_2(9.2) = 2.21916$, error 0.00004. The error estimate is

$$p_2(9.2) - p_1(9.2) = 0.00031.$$

(b) Extrapolation gives a much larger error. The difference table is

0.2	0.9980		
		−0.0294	
0.4	0.9686		−0.0949
		−0.1243	
0.6	0.8443		−0.1842
		−0.3085	
0.8	0.5358		−0.2273
		−0.5358	
1.0	0.0000		

The differences not shown are not needed. Taking $x = 0.6, 0.8, 1.0$ gives the best result. Newton's formula (14) with $r = 0.1/0.2 = 0.5$ gives

$$0.8443 + 0.5 \cdot (-0.3085) + \frac{0.5 \cdot (-0.5)}{2} \cdot (-0.2273) = 0.7185,$$

$$\epsilon = -0.0004.$$

Similarly, by taking $x = 0.4, 0.6, 0.8$ we obtain

$$0.9686 + 1.5 \cdot (-0.1243) + \frac{1.5 \cdot 0.5}{2} \cdot (-0.1842) = 0.7131, \qquad \epsilon = 0.0050.$$

Taking $x = 0.2, 0.4, 0.6$, we extrapolate and get a much poorer result:

$$0.9980 + 2.5 \cdot (-0.0294) + \frac{2.5 \cdot 1.5}{2} \cdot (-0.0949) = 0.7466, \qquad \epsilon = -0.0285.$$

(e) 0.386 4185, exact to 7S.

SECTION 17.4. Splines, page 861

Purpose. Interpolation of data $(x_0, f_0), \cdots, (x_n, f_n)$ by a (cubic) spline, that is, a twice continuously differentiable function that in each of the intervals $[x_0, x_1], [x_1, x_2], \cdots$ is given by a polynomial of third degree at most.

Short Courses. This section may be omitted.

Comments on Content

Higher order polynomials tend to oscillate between nodes—$P_8(x)$ in Fig. 402 is typical—and splines were introduced to avoid that phenomenon. This motivates their application.

If we impose the additional condition (3) with given k_0 and k_n, then for given data the cubic spline is unique.

SOLUTIONS TO PROBLEM SET 17.4, page 867

2. Writing $f(x_j) = f_j$, $f(x_{j+1}) = f_{j+1}$, $x - x_j = F$, $x - x_{j+1} = G$ we get (6) in the form

$$\begin{aligned} p_j(x) = {} & f_j c_j^2 G^2 (1 + 2c_j F) \\ & + f_{j+1} c_j^2 F^2 (1 - 2c_j G) \\ & + k_j c_j^2 F G^2 \\ & + k_{j+1} c_j^2 F^2 G. \end{aligned}$$

If $x = x_j$, then $F = 0$, so that because $c_j = 1/(x_{j+1} - x_j)$,

$$p_j(x_j) = f_j c_j^2 (x_j - x_{j+1})^2 = f_j.$$

Similarly, if $x = x_{j+1}$, then $G = 0$ and

$$p_j(x_{j+1}) = f_{j+1} c_j^2 (x_{j+1} - x_j)^2 = f_{j+1}.$$

This verifies (4). By differentiation,

$$\begin{aligned} p_j'(x) &= f_j c_j^2[2G(1 + 2c_jF) + 2c_jG^2] \\ &\quad + f_{j+1}c_j^2[2F(1 - 2c_jG) - 2c_jF^2] \\ &\quad + k_jc_j^2[G^2 + 2FG] \\ &\quad + k_{j+1}c_j^2[2FG + F^2]. \end{aligned}$$

If $x = x_j$, then $F = 0$ and in the first line

$$2G(1 + c_jG) = 2(x_j - x_{j+1})\left(1 + \frac{x_j - x_{j+1}}{x_{j+1} - x_j}\right) = 0.$$

There remains

$$p_j'(x_j) = k_jc_j^2(x_{j+1} - x_j)^2 = k_j.$$

Similarly, if $x = x_{j+1}$, then $G = 0$ and

$$\begin{aligned} p_j'(x_{j+1}) &= f_{j+1}c_j^2[2(x_{j+1} - x_j) - 2c_j(x_{j+1} - x_j)^2] \\ &\quad + k_{j+1}c_j^2(x_{j+1} - x_j)^2 \\ &= k_{j+1} \end{aligned}$$

because $[\cdots] = 0$. This verifies (5).

4. This is simple and straightforward.

6. a_{j2} can be seen from (7), and a_{j3} follows directly as indicated in the text after (14).

8. $p_2(x) = x^2$. $[f(x) - p_2(x)]' = 4x^3 - 2x = 0$ gives the points of maximum deviation $x = \pm 1/\sqrt{2}$ and by inserting this, the maximum deviation itself,

$$|f(1/\sqrt{2}) - p_2(1/\sqrt{2})| = |\tfrac{1}{4} - \tfrac{1}{2}| = \tfrac{1}{4}.$$

For the spline $g(x)$ we get, taking $x \geqq 0$,

$$[f(x) - g(x)]' = 4x^3 + 2x - 6x^2 = 0.$$

A solution is $x = 1/2$. The corresponding maximum deviation is

$$f(\tfrac{1}{2}) - g(\tfrac{1}{2}) = \tfrac{1}{16} - (-\tfrac{1}{4} + 2 \cdot \tfrac{1}{8}) = \tfrac{1}{16},$$

which is merely 25% of the previous value.

10. Since the third derivative of a cubic polynomial is constant and $g(x)$ consists of cubic polynomials, $g'''(x)$ is always piecewise constant. Since $g'''(x)$ is assumed to be continuous, $g'''(x) = M = const$ throughout the entire interval. By integration $p_j(x)'' = Mx + A_j$. Since $g''(x)$ is always continuous, $A_j = A = const$ for all j. This idea and two more integrations show that $g(x)$ is just one cubic polynomial throughout the whole interval.

12. $p_0(x) = 1 - 2(x + 2) + (x + 2)^3$, $p_1(x) = 5 + 10x + 6x^2 - 4x^3$

14. $p_0(x) = x^3$, $p_1(x) = 1 + 3(x - 1) + 3(x - 1)^2 - (x - 1)^3$,
$p_2(x) = 6 + 6(x - 2) - 2(x - 2)^3$

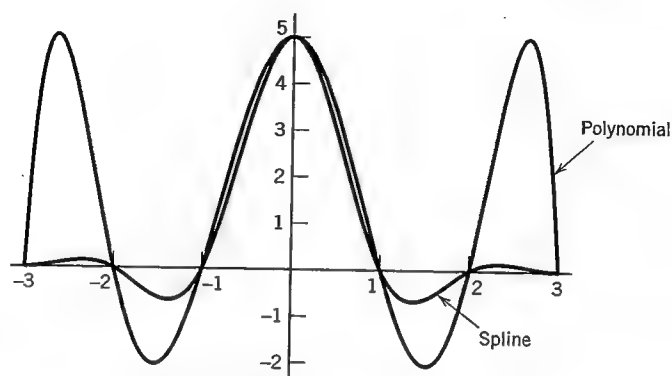
16. $p_0 = -\frac{3}{4}(x + 2)^2 + \frac{3}{4}(x + 2)^3 = 3 + 6x + \frac{15}{4}x^2 + \frac{3}{4}x^3$
$p_1 = \frac{3}{4}(x + 1) + \frac{3}{2}(x + 1)^2 - \frac{5}{4}(x + 1)^3 = 1 - \frac{9}{4}x^2 - \frac{5}{4}x^3$
$p_2 = 1 - \frac{9}{4}x^2 + \frac{5}{4}x^3$
$p_3 = -\frac{3}{4}(x - 1) + \frac{3}{2}(x - 1)^2 - \frac{3}{4}(x - 1)^3 = 3 - 6x + \frac{15}{4}x^2 - \frac{3}{4}x^3$.
The interpolation polynomial is (Fig. 405)

$$p(x) = 1 - \tfrac{5}{4}x^2 + \tfrac{1}{4}x^4.$$

18. $f(0) = 5$ (instead of 1) was chosen to avoid fractions $\frac{1}{5}$ throughout. Equation (12) gives $k_1 = -1$, $k_2 = 4$, $k_3 = 0$, $k_4 = -4$, $k_5 = 1$. From this and (13)–(14) we obtain

$$p_0 = (x + 3)^2 - (x + 3)^3 = -18 - 21x - 8x^2 - x^3$$
$$p_1 = -(x + 2) - 2(x + 2)^2 + 3(x + 2)^3 = 14 + 27x + 16x^2 + 3x^3$$
$$p_2 = 4(x + 1) + 7(x + 1)^2 - 6(x + 1)^3 = 5 - 11x^2 - 6x^3$$
$$p_3 = 5 - 11x^2 + 6x^3$$
$$p_4 = -4(x - 1) + 7(x - 1)^2 - 3(x - 1)^3 = 14 - 27x + 16x^2 - 3x^3$$
$$p_5 = x - 2 - 2(x - 2)^2 + (x - 2)^3 = -18 + 21x - 8x^2 + x^3.$$

Note that this is an even function, and so is the interpolation polynomial

$$p(x) = 5 - 6.80556x^2 + 1.94444x^4 - 0.138889x^6.$$

Section 17.4. Cubic spline versus polynomial of 6th degree in Problem 18

20. TEAM PROJECT.
(b) $x(t) = \frac{1}{2}t + \frac{5}{2}t^2 - 2t^3$, $y(t) = \frac{1}{2}t + (\frac{1}{4}\sqrt{3} - 1)t^2 + (\frac{1}{2} - \frac{1}{4}\sqrt{3})t^3$
(c) $x(t) = t + 2t^2 - 2t^3$, $y(t) = t + (\frac{1}{2}\sqrt{3} - 2)t^2 + (1 - \frac{1}{2}\sqrt{3})t^3$

SECTION 17.5. Numerical Integration and Differentiation, page 869

Purpose. Evaluation of integrals of empirical functions, functions not integrable by elementary methods, etc.

Main Content, Important Concepts

Simpson's rule (7) (most important), error (8), (10)
Trapezoidal rule (2), error (4), (5)
Gaussian integration
Adaptive integration with Simpson's rule (Example 6)
Numerical differentiation

Short Courses. Discuss and apply Simpson's rule.

Comments on Content

The range of numerical integration includes empirical functions, as measured or recorded in experiments, functions that cannot be integrated by the usual methods, or functions that can be integrated by those methods but lead to expressions whose computational evaluation would be more complicated than direct numerical integration of the integral itself.

Simpson's rule approximates the integrand by quadratic parabolas. Approximations by higher order polynomials are possible, but lead to formulas that are generally less practical.

Numerical differentiation can sometimes be avoided by changing the mathematical model of the problem.

SOLUTIONS TO PROBLEM SET 17.5, page 880

2. $A \leqq J \leqq B$, $A = h\Sigma A_j$, $B = h\Sigma B_j$, A_j and B_j being lower and upper bounds for f in the jth subinterval. $0.681 \leqq J \leqq 0.808$.

4. $h = 1$, $J_1 = 0.5$; $h = 0.5$, $J_{0.5} = 0.28125$, $\epsilon_{0.5} = \frac{1}{3}(0.28125 - 0.5) = -0.07292$ (actual error -0.08125); $h = 0.25$, $J_{0.25} = 0.22070$, $\epsilon_{0.25} = \frac{1}{3}(0.22070 - 0.28125) = -0.02018$ (actual error -0.02070). The agreement is very good. The same is true in Prob. 5, where we integrate a trigonometric function (instead of a single power of x).

6. $h|\frac{1}{2}\epsilon_0 + \epsilon_1 + \cdots + \epsilon_{n-1} + \frac{1}{2}\epsilon_n| \leqq [(b-a)/n]nu = (b-a)u$. This is similar to the corresponding proof for Simpson's rule given in the text.

8. 0.693150. Exact to 6D: $\ln 2 = 0.693147$

10. 0.07392 8162. Exact to 9D: 0.07392 8106

12. 0.78539 8153. Exact to 9D: 0.78539 8163

14. $C = -0.5^4/90$ in (9), $-0.000695 \leqq \epsilon \leqq -0.000094$ (actual error -0.000292). In (10),

$$\epsilon_{0.5} \approx \tfrac{1}{15}(0.864956 - 0.868951) = -0.000266.$$

Note that the absolute value of this is less than that of the actual error, and we must carefully distinguish between bounds and approximate values.

16. 0.946146, 0.946083. Exact to 6D: 0.946083. A modest table is included in Appendix 5. See Ref. [1] for larger tables.

18. 0.4716. Exact to 4D: 0.4615. For tables, see Ref. [1].

20. 0.91973 (exact to 5D). For a table, see the end of Ref. [A7], the standard book on Bessel functions.

22. **(a)** $M_2 = 2$, $M_2^* = 1/4$, hence by (4) and the accuracy requirement,

$$|KM_2| = \frac{2}{12n^2} = \frac{1}{2} \cdot 10^{-5},$$

which gives $n = 183$.

(b) From (9) with $f^{iv} = 24/x^5$, $M_4 = 24$, and the accuracy requirement,

$$|CM_4| = \frac{24}{180 \cdot (2m)^4} = \frac{1}{2} \cdot 10^{-5},$$

which gives $2m = 14$.

24. **TEAM PROJECT.** The factor $2^4 = 16$ comes in because we have replaced h by $\frac{1}{4}h$, giving for h^2 now $(\frac{1}{4}h)^2 = \frac{1}{16}h^2$. In the next step (with $h/8$) the error ϵ_{43} has the factor $1/(2^6 - 1) = \frac{1}{63}$, etc.

For $f(x) = e^{-x}$ the table of J and ϵ values is

$J_{11} = 1.135335$				
	$\epsilon_{21} = -0.066596$			
$J_{21} = 0.935547$		$J_{22} = 0.868951$		
	$\epsilon_{31} = -0.017648$		$\epsilon_{32} = -0.000266$	
$J_{31} = 0.882604$		$J_{32} = 0.864956$		$J_{33} = 0.864690$

J_{33} is exact to 4D.

For $f(x) = \frac{1}{4}\pi x^4 \cos \frac{1}{4}\pi x$ the Romberg table is

$J_{11} = 0$						
	$\epsilon_{21} = 0.185120$					
$J_{21} = 0.555360$		$J_{22} = 0.74048$				
	$\epsilon_{31} = 0.168597$		$\epsilon_{32} = 0.003262$			
$J_{31} = 1.06115$		$J_{32} = 1.22975$		$J_{33} = 1.26236$		
	$\epsilon_{41} = 0.049142$		$\epsilon_{42} = 0.001864$		$\epsilon_{43} = -0.00004$	
$J_{41} = 1.20857$		$J_{42} = 1.25771$		$J_{43} = 1.25958$		$J_{44} = 1.25953$

J_{44} is exact to 5D.

26. 0.240, which is not exact. It can be shown that the error term of the present formula is $h^3 f^{(4)}(\xi)/12$, whereas that of (15) is $h^4 f^{(5)}(\xi)/30$, where $x_2 - h < \xi < x_2 + h$. In our case this gives the exact value $0.240 + 0.016 = 0.256$ and $0.256 + 0 = 0.256$, respectively.

28. Differentiating (14) in Sec. 17.3 with respect to r and using $dr = dx/h$ we get

$$\frac{df(x)}{dr} = hf'(x) \approx \Delta f_0 + \frac{2r-1}{2!}\Delta^2 f_0 + \frac{3r^2 - 6r + 2}{3!}\Delta^3 f_0 + \cdots .$$

Now $x = x_0$ gives $r = (x - x_0)/h = 0$ and the desired formula follows.

SOLUTIONS TO CHAPTER 17 REVIEW, page 882

22. 0.14910×10^2, -0.91842×10^{-1}, 0.30303×10^4, -0.81818×10^{-1}, 0.97656×10^{-3}

24. 8.2586, 8.258, 9.90, impossible

26. $26.855 \leqq d \leqq 26.965$

28. In multiplication, relative errors add (see the proof of Theorem 1 in Sec. 17.1).

30. Multiply numerator and denominator by $\sqrt{x^2 + 16} + 4$, so that the given expression takes the form $x^2/(\sqrt{x^2 + 16} + 4)$.

32. Because $|g'(x)|$ is small (0.038) near the solution 0.739085.

34. 0.641714

36. 0.450184

38. 0.4, 0.085

40. 2.969

42. $3 + 3(x + 1) - 6(x + 1)^2 + 2(x + 1)^3$ if $-1 \leqq x \leqq 1$,
$1 + 3(x - 1) + 6(x - 1)^2 - (x - 1)^3$ if $1 \leqq x \leqq 3$,
$23 + 15(x - 3) - (x - 3)^3$ if $3 \leqq x \leqq 5$

44. $J_{0.5} = 0.90266$, $J_{0.25} = 0.90450$, $\epsilon_{0.25} = 0.00012$

CHAPTER 18 Numerical Methods in Linear Algebra

SECTION 18.1. Linear Systems: Gauss Elimination, page 886

Purpose. To explain the Gauss elimination, which is a solution method for linear systems of algebraic equations by systematic elimination (reduction to triangular form).

Main Content, Important Concepts

Gauss elimination, back substitution

Pivot equation, pivot, choice of pivot

Operations count, order [e.g., $O(n^3)$]

Comments on Content

This section is independent of Chap. 6 on matrices (in particular, Sec. 6.3, where the Gauss elimination is also considered).

Gauss's method and its variants (Sec. 18.2) are the most important solution methods for those systems (with matrices that do not have too many zeros).

The Gauss–Jordan method (Sec. 18.2) is less practical because it requires more operations than the Gauss elimination.

Cramer's rule (Sec. 6.6) would be totally impractical in numerical work, even for systems of modest size.

SOLUTIONS TO PROBLEM SET 18.1, page 893

2. $x_2 = (25/42)x_1$, x_1 arbitrary

4. $x_1 = 3.1$, $x_2 = -5.2$

6. $x_1 = 120$, $x_2 = 0.3$

8. $x_1 = 5.3$, $x_2 = 0$, $x_3 = -2.1$

10. $x_1 = -\frac{5}{8}x_3$, $x_2 = \frac{1}{8}x_3$, x_3 arbitrary; rank $\mathbf{A} = 2$

12. No solution; the matrix obtained at the end is

$$\begin{bmatrix} 5 & 3 & 1 & 2 \\ 0 & -4 & 8 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

14. $x_1 = -\frac{1}{12}$, $x_2 = \frac{7}{6}$, $x_3 = -\frac{5}{12}$

16. x_1 arbitrary, $x_2 = 3x_1 - 5$, $x_3 = -5x_1 + 14$; rank $\mathbf{A} = 2$

18. $x_1 = 4.2$, $x_2 = 0$, $x_3 = -1.8$, $x_4 = 2.0$

20. TEAM PROJECT. (a) (i) $a \neq 1$ to make $D = a - 1 \neq 0$; (ii) $a = 1$, $b = 3$; (iii) $a = 1$, $b \neq 3$.

(b) $x_1 = \frac{1}{2}(3x_3 - 1)$, $x_2 = \frac{1}{2}(-5x_3 + 7)$, x_3 arbitrary is the solution of the first system. The second system has no solution.

(c) $\det \mathbf{A} = 0$ can change to $\det \mathbf{A} \neq 0$ because of round-off.

(d) $(1 - 1/\epsilon)x_2 = 2 - 1/\epsilon$ eventually becomes $x_2/\epsilon \approx 1/\epsilon$, $x_2 = 1$, $x_1 = (1 - x_2)/\epsilon \approx 0$. The exact solution is $x_1 = 1/(1 - \epsilon)$, $x_2\ (1 - 2\epsilon)/(1 - \epsilon)$. We obtain it if we take $x_1 + x_2 = 2$ as the pivot equation.

(e) The exact solution is $x_1 = 1, x_2 = -4$. The 3-digit calculation gives $x_2 = -4.5$, $x_1 = 1.27$ without pivoting and $x_2 = -6$, $x_1 = 2.08$ with pivoting. This shows that 3S is simply not enough. The 4-digit calculations give $x_2 = -4.095$, $x_1 = 1.051$ without pivoting and the exact result $x_2 = -4$, $x_1 = 1$ with pivoting.

SECTION 18.2. Linear Systems: LU-Factorization, Matrix Inversion, page 894

Purpose. To discuss Doolittle's, Crout's, and Cholesky's methods, three methods for solving linear systems that are based on the idea of writing the coefficient matrix as a product of two triangular matrices ("LU-factorization"). Furthermore, we discuss matrix inversion by the Gauss–Jordan elimination.

Main Content, Important Concepts

Doolittle's and Crout's methods for arbitrary square matrices

Cholesky's method for positive definite symmetric matrices

Numerical matrix inversion

Short Courses. Doolittle's method and the Gauss–Jordan elimination.

Comment on Content

L suggests "lower triangular" and U "upper triangular." For Doolittle's method, these are the same as the matrix of the multipliers and of the triangular system in the Gauss elimination.

The point is that in the present methods, one solves one equation at a time, no systems.

SOLUTIONS TO PROBLEM SET 18.2, page 899

2. $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 6 \\ 0 & 4 \end{bmatrix}$, $\begin{aligned} x_1 &= 4.25 \\ x_2 &= -3.67 \end{aligned}$

4. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{aligned} x_1 &= -1 \\ x_2 &= 2 \\ x_3 &= -1 \end{aligned}$

6. $\begin{bmatrix} 1 & 0 & 0 \\ \frac{9}{5} & 1 & 0 \\ \frac{2}{5} & \frac{13}{61} & 1 \end{bmatrix}\begin{bmatrix} 5 & 9 & 2 \\ 0 & -\frac{61}{5} & -\frac{13}{5} \\ 0 & 0 & \frac{46}{61} \end{bmatrix}$, $\begin{aligned} x_1 &= 2 \\ x_2 &= 0 \\ x_3 &= 7 \end{aligned}$

8. $\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0.3 & 0.2 & 0.1 \end{bmatrix}\begin{bmatrix} 0.1 & 0 & 0.3 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}$, $\begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 4 \end{aligned}$

10. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & -1 & 3 & 0 \\ 2 & 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, $\begin{aligned} x_1 &= 2 \\ x_2 &= -3 \\ x_3 &= 4 \\ x_4 &= -1 \end{aligned}$

12. $\mathbf{x}^T(-\mathbf{A})\mathbf{x} = -\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$, no; $\mathbf{x}^T\mathbf{A}^T\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T\mathbf{x})^T$ because this is a scalar; this gives $\mathbf{x}^T\mathbf{A}^{TT}\mathbf{x}^{TT} = \mathbf{x}^T\mathbf{A}\mathbf{x} > 0$, yes. $\mathbf{A} + \mathbf{B}$ is positive definite; $\mathbf{A} - \mathbf{B}$ is not.

14. TEAM PROJECT. (a) The formulas for the entries of $\mathbf{L} = [l_{jk}]$ and $\mathbf{U} = [u_{jk}]$ are

$$l_{j1} = a_{j1} \qquad j = 1, \cdots, n$$

$$u_{1k} = \frac{a_{1k}}{l_{11}} \qquad k = 2, \cdots, n$$

$$l_{jk} = a_{jk} - \sum_{s=1}^{k-1} l_{js}u_{sk} \qquad j = k, \cdots, n; \quad k \geqq 2$$

$$u_{jk} = \frac{1}{l_{jj}}\left(a_{jk} - \sum_{s=1}^{j-1} l_{js}u_{sk}\right) \qquad k = j + 1, \cdots, n; \quad j \geqq 2.$$

(b) $\begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix}, \begin{matrix} x_1 = 4 \\ x_2 = 3 \end{matrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 9 & 0 \\ 2 & 12 & 4 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix}, \begin{matrix} x_1 = 278 \\ x_2 = 41 \\ x_3 = -16.5 \end{matrix}$

(c) To get the Doolittle factorization, take the transpose of Crout's factorization. The Cholesky factorization is

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 3 & 0 \\ 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

16. $\begin{bmatrix} \frac{11}{2} & -5 & 1 \\ -7 & \frac{23}{3} & -\frac{5}{3} \\ 1 & -\frac{4}{3} & \frac{1}{3} \end{bmatrix}$

18. $\frac{1}{9}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \frac{1}{9}\mathbf{A}^T$. Hence $\frac{1}{3}\mathbf{A}$ is orthogonal.

20. $\det \mathbf{A} = 0$ as given, but rounding makes $\det \mathbf{A} \neq 0$ and may completely change the situation with respect to existence of solutions of linear systems, a point to be watched for when using a CAS. In the present case we get (a) -0.00000035, (b) -0.00001998, (c) -0.00028189, (d) 0.002012, (e) 0.0002.

SECTION 18.3. Linear Systems: Solution by Iteration, page 900

Purpose. To familiarize the student with the idea of solving linear systems by iteration, to explain in what situations that is practical, and to discuss the most important method (Gauss–Seidel iteration) and its convergence.

Main Content, Important Concepts

Distinction between direct and indirect methods

Gauss–Seidel iteration, its convergence, its range of applicability

Matrix norms

Jacobi iteration

Short Courses. Gauss–Seidel iteration only.

Comments on Content

The Jacobi iteration appeals by its simplicity but is of no practical value.

A word on the frequently occurring sparse matrices may be good. For instance, we have about 99.5% zeros in solving the Laplace equation in two dimensions by using a 1000×1000 grid and the usual five-point pattern (Sec. 19.4).

SOLUTIONS TO PROBLEM SET 18.3, page 905

2. The exact solution 3, −9, 6 is reached at Step 8, rather quickly due to the fact that the spectral radius of **C** is 0.125, hence rather small.

4. Interchange the first equation and the last equation. Then the exact solution −2.5, 2, 4.5 is reached at Step 11, the spectral radius of **C** being $1/\sqrt{15} = 0.258199$. (The eigenvalues are complex conjugates, and the third eigenvalue is 0, as always for the present **C**.)

6. The exact solution is 2, 0, 1. Step 10 gives $[2.00144 \quad -0.00221311 \quad 0.999779]^T$. The spectral radius $(\frac{2}{3})^{3/2} = 0.544331$ of **C** is relatively large.

8. In (a) we obtain

$$\mathbf{C} = -(\mathbf{I} + \mathbf{L})^{-1}\mathbf{U}$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ -0.1 & 1 & 0 \\ -0.09 & -0.1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0.1 & 0.1 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.100 & -0.100 \\ 0 & 0.010 & -0.090 \\ 0 & 0.009 & 0.019 \end{bmatrix}$$

and $\|\mathbf{C}\| = 0.2 < 1$ by (11), which implies convergence by (8).

In (b) we have

$$\begin{bmatrix} 1 & 1 & 10 \\ 10 & 1 & 1 \\ 1 & 10 & 1 \end{bmatrix} = (\mathbf{I} + \mathbf{L}) + \mathbf{U}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 10 & 1 & 0 \\ 1 & 10 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 10 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From this we compute

$$\mathbf{C} = -(\mathbf{I} + \mathbf{L})^{-1}\mathbf{U} = -\begin{bmatrix} 1 & 0 & 0 \\ -10 & 1 & 0 \\ 99 & -10 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 10 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 1 & 10 \\ 0 & -10 & -99 \\ 0 & 99 & 980 \end{bmatrix}.$$

Developing the characteristic determinant by its first column, we obtain

$$\lambda \begin{vmatrix} -\lambda - 10 & -99 \\ 99 & -\lambda + 980 \end{vmatrix} = \lambda(\lambda^2 - 970\lambda + 1),$$

which shows that one of the eigenvalues is greater than 1 in absolute value, so that we have divergence.

10. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5.5 \\ -10.75 \\ 8.5 \end{bmatrix}, \begin{bmatrix} 2.5625 \\ -7.75 \\ 5.5625 \end{bmatrix}, \begin{bmatrix} 3.3125 \\ -9.21875 \\ 6.3125 \end{bmatrix}, \begin{bmatrix} 2.94531 \\ -8.84375 \\ 5.94531 \end{bmatrix}, \begin{bmatrix} 3.03906 \\ -9.02734 \\ 6.03906 \end{bmatrix}.$

Step 5 of the Gauss–Seidel iteration gives the better result

$$[2.99969 \quad -9.00015 \quad 5.99996]^T. \qquad \text{Exact: } [3 \quad -9 \quad 6]^T.$$

12. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1.8125 \\ 2.58333 \\ 1.7 \end{bmatrix}, \begin{bmatrix} -2.29583 \\ 2.81875 \\ 3.95 \end{bmatrix}, \begin{bmatrix} -2.63594 \\ 2.14931 \\ 4.33667 \end{bmatrix}, \begin{bmatrix} -2.51691 \\ 2.07710 \\ 4.60875 \end{bmatrix}, \begin{bmatrix} -2.53287 \\ 1.96657 \\ 4.51353 \end{bmatrix}$

Step 5 of the Gauss–Seidel iteration gives the more accurate result

$$[-2.49475 \quad 1.99981 \quad 4.49580]^T. \qquad \text{Exact: } [-2.5 \quad 2 \quad 4.5]^T.$$

14. The eigenvalues of $\mathbf{I} - \mathbf{A}$ are 0.5, 0.5, −1. Here, $\mathbf{A}$ is $\frac{1}{2}$ times the coefficient matrix of the given system.

16. $\sqrt{52} = 7.2, 6, 6$ **18.** $3a, 3a, 3a$ **20.** $\sqrt{300} = 17.32, 10, 10$

SECTION 18.4. Linear Systems: Ill-Conditioning, Norms, page 906

Purpose. To discuss ill-conditioning quantitatively in terms of norms, leading to the condition number and its role in judging the effect of inaccuracies on solutions.

Main Content, Important Concepts

Ill-conditioning, well-conditioning

Symptoms of ill-conditioning

Residual

Vector norms

Matrix norms

Condition number

Effect of inaccuracies of coefficients on solutions

Comment on Content

Reference [E8] in Appendix 1 gives some help when $\mathbf{A}^{-1}$, needed in $\kappa(\mathbf{A})$, is unknown (as is usual in practice).

SOLUTIONS TO PROBLEM SET 18.4, page 912

2. $12, \sqrt{50} \approx 7.07, 5, [0.6 \quad 0.8 \quad -1]$

4. $5, \sqrt{5} \approx 2.24, 1, [1 \quad 1 \quad 1 \quad 1 \quad 1]$

6. $1, 1, 1, [0 \quad 0 \quad 0 \quad 1 \quad 0]$

8. 2, 2; $2 \cdot 2 = 4$, $2 \cdot 2 = 4$. Inverse

$$\begin{bmatrix} -0.75 & 1.25 \\ 1.25 & -0.75 \end{bmatrix}$$

10. 5.5, 5.5; $5.5 \cdot 136 = 748$, $5.5 \cdot 136 = 748$, ill-conditioned. Inverse

$$\begin{bmatrix} 10 & -60 & 60 \\ 3 & -12 & 10 \\ -12 & 64 & -60 \end{bmatrix}$$

12. 19, 21; $19 \cdot 13 = 247$, $21 \cdot 13 = 273$. Inverse

$$\begin{bmatrix} 6 & 4 & 3 \\ 4 & 3 & 2 \\ 3 & 4 & 2 \end{bmatrix}$$

14. $x_1 = 1$, $x_2 = 1$; $x_1 = 0.845455$, $x_2 = 1.27273$ (6S); $\kappa(\mathbf{A}) = 4.7 \cdot 42.7273 = 200.8$

16. The residual is $[0.145 \quad 0.120]^T$, whereas the approximate solution deviates from the true solution by a factor 5 (the first component) and $3\frac{1}{2}$. This is a consequence of the fact that the system is very ill-conditioned.

18. By (12), $1 = \|\mathbf{I}\| = \|\mathbf{A}\mathbf{A}^{-1}\| \leqq \|\mathbf{A}\| \; \|\mathbf{A}^{-1}\| = \kappa(\mathbf{A})$. For the Frobenius norm, $\sqrt{n} = \|\mathbf{I}\| \leqq \kappa(\mathbf{A})$.

20. TEAM PROJECT. (a) Formula (18a) is obtained from

$$\max |x_j| \leqq \Sigma\, |x_k| = \|\mathbf{x}\|_1 \leqq n \max |x_j| = n\|\mathbf{x}\|_\infty.$$

Equation (18b) follows from (18a) by division by n.

(b) To get the first inequality in (19a) consider the square of both sides and then take square roots on both sides. The second inequality in (19a) follows by means of the Cauchy–Schwarz inequality and a little trick worth remembering,

$$\sum |x_j| = \sum 1 \cdot |x_j| \leqq \sqrt{\sum 1^2}\,\sqrt{\sum |x_j|^2} = \sqrt{n}\,\|\mathbf{x}\|_2$$

To get (19b), divide (19a) by $\sqrt{n}$.

(c) Let $\mathbf{x} \neq \mathbf{0}$. Set $\mathbf{x} = \|\mathbf{x}\|\mathbf{y}$. Then $\|\mathbf{y}\| = \|\mathbf{x}\|/\|\mathbf{x}\| = 1$. Also, $\mathbf{A}\mathbf{x} = \mathbf{A}(\|\mathbf{x}\|\mathbf{y}) = \|\mathbf{x}\|\,\mathbf{A}\mathbf{y}$ since $\|\mathbf{x}\|$ is a number. Hence $\|\mathbf{A}\mathbf{x}\|/\|\mathbf{x}\| = \|\mathbf{A}\mathbf{y}\|$, and in (9), instead of taking the maximum over *all* $\mathbf{x} \neq \mathbf{0}$, since $\|\mathbf{y}\| = 1$ we only take the maximum over all $\mathbf{y}$ of norm 1. Write $\mathbf{x}$ for $\mathbf{y}$ to get (10) from this.

(d) These "axioms of a norm" follow from (3), which are the axioms of a vector norm.

SECTION 18.5. Method of Least Squares, page 914

Purpose. To explain Gauss's least squares method of "best fit" of straight lines to given data $(x_0, y_0), \cdots, (x_n, y_n)$ and its extension to best fit of quadratic polynomials, etc.

Main Content, Important Concepts

Least squares method

Normal equations (4) for straight lines

Normal equations (8) for quadratic polynomials

Short Courses. Discuss the linear case only.

Comment. Normal equations are often ***ill-conditioned,*** so that results may be sensitive to round-off. For another (theoretically much more complicated) method, see Ref. [E3], p. 201.

SOLUTIONS TO PROBLEM SET 18.5, page 916

2. $3.68 - 1.22x$. Note the considerable change of the slope.

4. $95.26 - 0.574t$, where $t = 0$ [min] corresponds to 12:00. This is a cooling process following Newton's law of cooling, an exponential decrease of temperature; this explains the better fit in Prob. 5.

6. $\Sigma x_j = 2950$, $\Sigma x_j^2 = 1\,822\,500$, $\Sigma y_j = 7010$, $\Sigma x_j y_j = 4\,490\,000$; this gives the normal equations

$$\begin{aligned} 5\,b_0 + \quad 2\,950\,b_1 &= \quad 7\,010 \\ 2950\,b_0 + 1\,822\,500\,b_1 &= 4\,490\,000. \end{aligned}$$

The solution is $b_0 = -1145.79$, $b_1 = 4.32$. *Answer:*

$$y = -1145.79 + 4.32x.$$

8. $s(F) = 0.033 + 0.314F$, $k = F/s = 1/0.314 = 3.185$

10. $0.955 - 1.159x + 0.932x^2$

12. $1660 + 656x - 32x^2$

14. $5.9 - 0.05x$; $5.9 - 0.95x + 0.23x^2$

16.
$$\begin{aligned} b_0 n \quad + b_1\Sigma x_j \ + b_2\Sigma x_j^2 + b_3\Sigma x_j^3 &= \Sigma y_j \\ b_0\Sigma x_j \ + b_1\Sigma x_j^2 + b_2\Sigma x_j^3 + b_3\Sigma x_j^4 &= \Sigma x_j y_j \\ b_0\Sigma x_j^2 + b_1\Sigma x_j^3 + b_2\Sigma x_j^4 + b_3\Sigma x_j^5 &= \Sigma x_j^2 y_j \\ b_0\Sigma x_j^3 + b_1\Sigma x_j^4 + b_2\Sigma x_j^5 + b_3\Sigma x_j^6 &= \Sigma x_j^3 y_j \end{aligned}$$

18. $-0.15 + 0.35x$; $0.09 + 0.35x - 0.14x^2$; $-0.03 - 1.54x - 0.11x^2 + 0.57x^3$. Note the large x^3-term, which had to be expected from the position of the given points.

20. TEAM PROJECT. (a) We substitute $F_m(x)$ into the integral and perform the square. This gives

$$\|f - F_m\|^2 = \int_a^b f^2\,dx - 2\sum_{j=0}^{m} a_j \int_a^b f y_j\,dx + \sum_{j=0}^{m}\sum_{k=0}^{m} a_j a_k \int_a^b y_j y_k\,dx.$$

This is a quadratic function in the coefficients. We take the partial derivative with respect to any one of them, call it a_l, and equate this derivative to zero. This gives

$$0 - 2\int_a^b f y_l\,dx + 2\sum_{j=0}^{m} a_j \int_a^b y_j y_l\,dx = 0.$$

Dividing by 2 and taking the first integral to the right gives the system of normal equations, with $l = 0, \cdots, m$.

(b) In the case of a polynomial we have

$$\int_a^b y_j y_l\,dx = \int_a^b x^{j+l}\,dx,$$

which can be readily integrated. In particular, if $a = 0$ and $b = 1$, integration from 0 to 1 gives $1/(j + l + 1)$, and we obtain the **Hilbert matrix** as the coefficient matrix.

(c) In the case of an orthogonal system we see from (4), Sec. 4.8, with $p(x) = 1$ (as for the Legendre polynomials, or with any weight function $p(x)$ corresponding to the given system) and l instead of m that $a_l = b_l/\|y_l\|^2$.

SECTION 18.6. Matrix Eigenvalue Problems: Introduction, page 917

Purpose. This section is a collection of concepts and a handful of theorems on matrix eigenvalues and eigenvectors that are frequently needed in numerical methods; some of them will be discussed in the remaining sections of the chapter and others can be found in more advanced or more specialized books listed in part E of Appendix 1.

The section frees both the instructor and the student from the task of locating these matters in Chaps. 6 and 7, which contain much more material and should be consulted only if problems on one or the other matters are wanted (depending on the background of the student) or if a proof might be of interest.

SECTION 18.7. Inclusion of Matrix Eigenvalues, page 920

Purpose. To discuss theorems that give approximate values and error bounds of eigenvalues of general (square) matrices (Theorems 1, 2, 4, Example 2) and of special matrices (Theorem 6).

Main Content, Important Concepts

Gerschgorin's theorem (Theorem 1)

Sharpened Gerschgorin's theorem (Theorem 2)

Gerschgorin's theorem improved by similarity (Example 2)

Strict diagonal dominance (Theorem 3)

Schur's inequality (Theorem 4), normal matrices

Perron's Theorem (Theorem 5)

Collatz's theorem (Theorem 6)

Short Courses. Discuss Theorems 1 and 6.

Comments on Content

It is important to emphasize that one must always make sure whether or not a thoerem applies to a given matrix. Some theorems apply to any real or complex square matrices whatsoever, whereas others are restricted to certain classes of matrices.

The exciting Gerschgorin's theorem was one of the early theorems on numerical methods for eigenvalues; it appeared in *Bull. Acad. Sciences de l'URSS* (Classe mathém, 7-e série, Leningrad, 1931, p. 749), and shortly thereafter in the German *Zeitschrift für angewandte Mathematik und Mechanik.*

SOLUTIONS TO PROBLEM SET 18.7, page 924

2. Symmetric matrix; hence we get intervals on the real axis, $9.7 \leqq \lambda \leqq 10.3$, $5.9 \leqq \lambda \leqq 6.1$, $2.8 \leqq \lambda \leqq 3.2$. The eigenvalues (6S-values) are 10.0082, 5.99751, 2.99429.

4. i, 0, $3 + 4i$, radii $0.5 + \sqrt{2}$, $\sqrt{2} + \sqrt{5}$, 2.
Spectrum (6S-values) $-0.0933282 + 1.061i$, $-0.403385 - 0.72446i$, $3.49671 + 4.66346i$

6. 0, 0.2, 1.2, radii 1.3, 2, 0.1, or, by taking the transpose, centers as before, radii 1.1, 0.5, 1.8, and we can take the smaller of the two in each case.
Spectrum $0.108609 \pm 0.742484i$ (absolute value 0.750386), 1.182781.

8. $\mathbf{T}$ with $t_{11} = t_{22} = 1$, $t = 34$ gives

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{34} \end{bmatrix} \begin{bmatrix} 10 & 0.1 & -0.2 \\ 0.1 & 6 & 0 \\ -0.2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 34 \end{bmatrix} = \begin{bmatrix} 10 & 0.1 & -6.8 \\ 0.1 & 6 & 0 \\ -\frac{0.2}{34} & 0 & 3 \end{bmatrix}.$$

Note that the disk with center 3 is still disjoint from that with center 10.

10. An example is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues are -1 and 1, so that the entire spectrum lies on the circle. A similar-looking 3×3 matrix or 4×4 matrix, etc., can be constructed with some or all of its eigenvalues on the circle.

12. This is a *"continuity proof."* Let $S = D_1 \cup D_2 \cup \cdots \cup D_p$ without restriction, where D_j is the Gerschgorin disk with center a_{jj}. We write $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where $\mathbf{B} = \text{diag}\,(a_{jj})$ is the diagonal matrix with the main diagonal of $\mathbf{A}$ as its diagonal. We now consider

$$\mathbf{A}_t = \mathbf{B} + t\mathbf{C} \qquad \text{for } 0 \leqq t \leqq 1.$$

Then $\mathbf{A}_0 = \mathbf{B}$ and $\mathbf{A}_1 = \mathbf{A}$. Now by algebra, the roots of the characteristic polynomial $f_t(\lambda)$ of $\mathbf{A}_t$ (that is, the eigenvalues of $\mathbf{A}_t$) depend continuously on the coefficients of $f_t(\lambda)$, which in turn depend continuously on t. For $t = 0$, the eigenvalues are $a_{11}, \cdots, a_{nn}$. If we let t increase continuously from 0 to 1, the eigenvalues move continuously and, by Theorem 1, for each t lie in the Gerschgorin disks with centers a_{jj} and radii

$$tr_j \qquad \text{where} \qquad r_j = \sum_{k \neq j} |a_{jk}|.$$

Since at the end, S is disjoint from the other disks, the assertion follows.

14. These proofs follow readily from the definition of these classes of matrices.

16. $\mathbf{A}^2(\mathbf{A}^2)^\mathsf{T} = \mathbf{AAA}^\mathsf{T}\mathbf{A}^\mathsf{T} = \mathbf{AA}^\mathsf{T}\mathbf{AA}^\mathsf{T} = \cdots \mathbf{A}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{AA}$; yes. $(\mathbf{AB})(\mathbf{AB})^\mathsf{T} = (\mathbf{AB})^\mathsf{T}(\mathbf{AB})$ if and only if $\mathbf{B}^\mathsf{T}\mathbf{A} = \mathbf{AB}^\mathsf{T}$; no, in general. $\mathbf{CC}^\mathsf{T} - \mathbf{C}^\mathsf{T}\mathbf{C}$ is symmetric, hence normal.

18. $29 \leqq \lambda \leqq 37$. It is interesting that the second starting vector gives the same interval. The third gives $31.66 \leqq \lambda \leqq 33.00$.

In practice, one would compute several steps and use the last two vectors for determining an interval that contains an eigenvalue. See Example 4 in the text.

20. CAS PROJECT. (a) The midpoint is an approximation for which the endpoints give error bounds.

(b) Nonmonotone behavior may occur if by chance you pick an initial vector close to an eigenvector corresponding to an eigenvalue that is not largest in absolute value.

SECTION 18.8. Eigenvalues by Iteration (Power Method), page 925

Purpose. Explanation of the power method for determining approximations and error bounds for eigenvalues of real symmetric matrices.

Main Content, Important Concepts

The iteration process of the power method

Rayleigh quotient (the approximate value)

Improvement of convergence by a spectral shift

Scaling (for eigenvectors)

Short Courses. Omit spectral shift.

Comments on Content

The method is simple but converges slowly, in general.

Symmetry of the matrix is essential to the validity of the error bound (1). The method as such can be applied to more general matrices.

SOLUTIONS TO PROBLEM SET 18.8, page 928

2. $\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 33 \\ 1 \end{bmatrix}, \begin{bmatrix} 195 \\ -101 \end{bmatrix}$; $q = -1, 2.76471, 5.81101$; $|\epsilon| \leqq 4, 4.94118, 3.23670$.

This illustrates that the error bounds ϵ need not be a monotone function of the step. They are large. This indicates that we are still far away from an eigenvalue (7 in the present case).

4. $q = 11.3333, \quad 11.9802, \quad 11.9994; \quad |\epsilon| \leqq 2.4944, \quad 0.4446, \quad 0.0742.$

The rapid convergence to the absolutely largest eigenvalue, 12, results from the fact that the other eigenvalues, 2 and -2, are much smaller in absolute value.

6. $q = 10.5000, \quad 11.1303, \quad 11.1831; \quad |\epsilon| \leqq 2.95804, \quad 1.36886, \quad 0.96374$

8. We get the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 8 \\ 14 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} 88 \\ 172 \\ 144 \end{bmatrix}$$

and from them the following. From the first two, Collatz gives $8 \leqq \lambda \leqq 14$, thus, if one wishes, the approximation 11 and error bound 3. Our Theorem 1 gives $q = 11.33$ (a bit closer to the exact $\lambda = 12$) and $|\epsilon| \leqq 2.5$, which is of the same order of magnitude as Collatz's bound.

From the second and third vectors, Collatz gives $11 \leqq \lambda \leqq 12.286$, say, the approximation 11.643 and error bound 0.643. Theorem 1 gives $q = 11.98$, which is much closer to 12, and $|\epsilon| \leqq 0.45$, about of the same quality as the bound by Collatz.

Remember that Collatz assumes positivity of the matrix entries, whereas in Theorem 1 we require symmetry of the matrix; in that sense the two theorems are not comparable. Theorem 1 uses all components of the vectors involved, and that tends to give better results than those from methods that use only one or two components.

Note further that Theorem 1 requires more operations (not excessively many, however).

10. The eigenvalues are $\lambda = \pm 5$. Corresponding eigenvectors are

$$\mathbf{z}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{z}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

and I have chosen $\mathbf{x}_0$ as

$$\mathbf{x}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{z}_1 + \mathbf{z}_2,$$

so that

$$\mathbf{x}_1 = 5(\mathbf{z}_1 - \mathbf{z}_2) = [5 \quad 15]^{\mathsf{T}}$$
$$\mathbf{x}_2 = 25(\mathbf{z}_1 + \mathbf{z}_2),$$

etc. From this,

$$\mathbf{x}_0^{\mathsf{T}}\mathbf{x}_1 = 0$$

and for the error bound we get

$$\sqrt{\frac{\mathbf{x}_1^{\mathsf{T}}\mathbf{x}_1}{\mathbf{x}_0^{\mathsf{T}}\mathbf{x}_0} - q^2} = \sqrt{\frac{250}{10} - 0} = 5,$$

and similarly in all the further steps. This shows that our error bound is the best possible in general.

12. The scaled vectors

$$\begin{bmatrix} -0.6 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.030303 \end{bmatrix}, \begin{bmatrix} 1 \\ -0.517949 \end{bmatrix}, \begin{bmatrix} 1 \\ -0.260014 \end{bmatrix}, \begin{bmatrix} 1 \\ -0.365776 \end{bmatrix}$$

approach their limit $[1 \quad -1/3]^{\mathsf{T}}$ (corresponding to $\lambda = 7$) in a somewhat irregular way during these first steps, indicating that the sequence begins with a linear combination of the two eigenvectors with a substantial contribution of each. The other eigenvector is $[1/3 \quad 1]^{\mathsf{T}}$, corresponding to $\lambda = -3$.

14. The eigenvalues are 11.2321, 4.28275, 0.44156, −7.95637, so the speed of convergence is determined by the ratio 11:8, approximately. The approximations obtained are

$$\begin{bmatrix} 0.466667 \\ 1 \\ 0.6 \\ 0.733333 \end{bmatrix}, \quad \begin{bmatrix} 0.57047 \\ 1 \\ 0.651007 \\ 0.973154 \end{bmatrix}, \quad \begin{bmatrix} 0.494303 \\ 1 \\ 0.582203 \\ 0.798155 \end{bmatrix}.$$

SECTION 18.9. Tridiagonalization and QR-Factorization, page 929

Purpose. Explanation of an optimal method for determining the whole spectrum of a real symmetric matrix by first reducing the matrix to a tridiagonal matrix with the same spectrum and then applying the QR-method, an iteration in which each step consists of a factorization (5) and a multiplication (6).

Comment on Content

Householder steps correspond to similarity transformations; hence the spectrum is preserved. The same holds for QR.

SOLUTIONS TO PROBLEM SET 18.9, page 937

2. $\mathbf{v} = [0 \quad 0.92388 \quad -0.382683]^T$, hence

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.707107 & 0.707107 \\ 0 & 0.707107 & 0.707107 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 6 & 5.09117 & 0 \\ 5.09117 & 9.6 & 0 \\ 0 & 0 & -0.8 \end{bmatrix}.$$

4. $$\mathbf{B} = \begin{bmatrix} 6 & -4.12311 & 0 & 0 \\ -4.12311 & 8.70588 & 7.24175 & 0 \\ 0 & 7.24175 & 2.18908 & 3.87609 \\ 0 & 0 & 3.87609 & 7.10504 \end{bmatrix}$$

6. $$\mathbf{B} = \begin{bmatrix} 5 & -4.24264 & 0 & 0 \\ -4.24264 & 6 & 1.41421 & 0 \\ 0 & 1.41421 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

8. $$\begin{bmatrix} 14.2004 & 0.0444 & 0 \\ 0.0444 & -6.3046 & -0.0668 \\ 0 & -0.0668 & 2.1042 \end{bmatrix}, \quad \begin{bmatrix} 14.2005 & -0.0197 & 0 \\ -0.0197 & -6.3052 & 0.0223 \\ 0 & 0.0223 & 2.1047 \end{bmatrix},$$

$$\begin{bmatrix} 14.2005 & 0.00875 & 0 \\ 0.00875 & -6.30524 & -0.00744 \\ 0 & -0.00744 & 2.10475 \end{bmatrix}.$$

6S-values of the eigenvalues are 14.2005, −6.30525, 2.10476. Hence the diagonal entries are more accurate than one would expect by looking at the size of the off-diagonal entries.

10. $$\begin{bmatrix} 0.6463 & -0.1471 & 0 \\ -0.1471 & 0.4201 & 0.0630 \\ 0 & 0.0630 & 0.1036 \end{bmatrix}, \quad \begin{bmatrix} 0.6988 & -0.0848 & 0 \\ -0.0848 & 0.3804 & 0.0152 \\ 0 & 0.0152 & 0.0908 \end{bmatrix},$$

$$\begin{bmatrix} 0.7145 & -0.0443 & 0 \\ -0.0443 & 0.3655 & 0.0037 \\ 0 & 0.0037 & 0.0901 \end{bmatrix}.$$

The spectrum is 0.72, 0.36, 0.09.

SOLUTIONS TO CHAPTER 18 REVIEW, page 938

16. $x_1 = 4, x_2 = 2x_3$ **18.** $x_1 = 3, x_2 = 3x_3 + 2$ **20.** $x_1 = 4, x_2 = -1, x_3 = 2$

22. All the entries of the triangular matrices are 1.

24. $$\frac{1}{226}\begin{bmatrix} 48 & -8 & -6 \\ -8 & 39 & 1 \\ -6 & 1 & 29 \end{bmatrix}$$

26. $$\begin{bmatrix} 10 & -10 & -10 \\ -2.99240 & 3.18590 & 2.95784 \\ -5.66690 & 5.96406 & 5.97097 \end{bmatrix}$$

28. Reorder to get convergence. Equation 1 becomes 2, 2 becomes 3, 3 becomes 1. Solution $x_1 = -2$, $x_2 = 8$, $x_3 = -1$. The iteration gives

$$\begin{bmatrix} -1.50667 \\ 8.17533 \\ -1.08464 \end{bmatrix}, \quad \begin{bmatrix} -2.01927 \\ 7.99250 \\ -0.996747 \end{bmatrix}, \quad \begin{bmatrix} -1.99925 \\ 8.00029 \\ -1.00013 \end{bmatrix}.$$

30. 2, $\sqrt{2}$, 1

32. 24, $\sqrt{136}$, 8

34. 2.4, $\sqrt{2.48}$, 1.2

36. 11

38. 9.1

40. $8.8 \cdot 19.15 = 168.5$. The matrix is ill-conditioned.

42. $y = 2.89 + 0.505x$

44. $y = 1.95 - 2.217x + 1.067x^2$

CHAPTER 19 Numerical Methods for Differential Equations

Major Changes

These include automatic variable step size selection in modern codes, the discussion of the Runge-Kutta-Fehlberg method, and the extension of Euler and Runge-Kutta methods to systems and higher order equations.

SECTION 19.1. Methods for First-Order Differential Equations, page 942

Purpose. To explain three numerical methods for solving initial value problems $y' = f(x, y)$, $y(x_0) = y_0$ by stepwise computing approximations to the solution at $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, etc.

Main Content, Important Concepts

Euler's method (3)

Automatic variable step size selection

Improved Euler method (7)

Classical Runge–Kutta method (Table 19.4)

Error and step size control

Runge–Kutta–Fehlberg method

Comments on Content

Euler's method is good for explaining the principle but is too crude to be of practical value.

The improved Euler method is a simple case of a predictor–corrector method.

The classical Runge–Kutta method is of order h^4 and is of great practical importance.

Principles for a good choice of h are important in any method.

f in the equation must be such that the problem has a unique solution (see Sec. 1.9).

SOLUTIONS TO PROBLEM SET 19.1, page 951

2. $y = \sin \frac{1}{2}\pi x$. Since the values obtained give $y_9 = 1.01170 > 1$, y_{10} comes out complex and is meaningless.

x_n	y_n	Error $\times 10^5$
0.1	0.15708	−65
0.2	0.31221	−319
0.3	0.46144	−745
0.4	0.60079	−1301
0.5	0.72636	−1926
0.6	0.83433	−2531
0.7	0.92092	−2991
0.8	0.98214	−3109
0.9	1.01170	−2401
1.0	—	—

4. $y = \tan x - x$ (special Riccati equation; set $y + x = u$, then $u' = u^2 + 1$, etc.). The computation gives

x_n	y_n	$y(x_n)$	Error $\times 10^6$
0.1	0.000 000	0.000 335	335
0.2	0.001 000	0.002 710	1 710
0.3	0.005 040	0.009 336	4 296
0.4	0.014 345	0.022 793	8 448
0.5	0.031 513	0.046 302	14 789
0.6	0.059 764	0.084 137	24 373
0.7	0.103 292	0.142 288	38 996
0.8	0.167 820	0.229 639	61 818
0.9	0.261 488	0.360 158	98 670
1.0	0.396 393	0.557 408	161 014

6. $y = 1/(1 + e^{-x})$. The given Verhulst equation is a special Bernoulli equation; see Sec. 1.6.

x_n	y_n	Error $\times 10^6$
0.1	0.524969	10
0.2	0.549813	21
0.3	0.574411	32
0.4	0.598645	42
0.5	0.622407	53
0.6	0.645593	63
0.7	0.668114	74
0.8	0.689890	84
0.9	0.710855	94
1.0	0.730955	104

8. $y = \tan 2x$. Note that the error is first negative and then positive and rapidly increasing, due to the behavior of the tangent.

x_n	y_n	Error $\times 10^5$
0.05	0.10050	−17
0.10	0.20304	−33
0.15	0.30981	−48
0.20	0.42341	−62
0.25	0.54702	−72
0.30	0.68490	−76
0.35	0.84295	−66
0.40	1.02989	−25
0.45	1,25930	+86
0.50	1.55379	362

10. The error of $y(1)$ is -0.0036, hence comparable to that in Prob. 7. The error of $y(2)$ is $+0.0067$, hence twice that in Prob. 7 and of the opposite sign.

12. $y = 0, 0.2055, 0.4276, 0.6587, 0.8924$; error $0, 0.0101, 0.0221, 0.0322, 0.0409$. Hence the error is about 20% less.

14. $y = 0, 0.1033, 0.2134, 0.3280, 0.4456, 0.5650, 0.6852, 0.8058, 0.9261$; error $0, 0.0011, 0.0022, \cdots, 0.0071$; about 15% of that in Prob. 11.

16. For instance, $x = 0.5$, $y = 0.632\,114\,762$ (error $0.58 \cdot 10^{-5}$); $x = 1$, $y = 0.864\,660\,452$ (error $0.43 \cdot 10^{-5}$); hence the error is substantially less, and it is interesting that it is not increasing: for $x = 0.1, \cdots, 1.0$, it is 0.26, 0.42, 0.52, 0.57, 0.58, 0.57, 0.54, 0.51, 0.47, 0.43 times 10^{-5}.

18. Errors $-0.004, -0.008, -0.011, -0.015, -0.017, -0.019, -0.021, -0.021, -0.021, -0.021$. For the improved Euler method, the errors times 10^5 are 0.8, 1.6, 2.2, 2.5, 2.5, 1.8, +0.4, −1.9, −5.1, −9.3.

20.

x_n	y_n	Error Estimate (10) × 10^9	Error × 10^9
0.1	1.20033 46725	3.0	−0.4
0.2	1.40271 00374	3.7	−1.9
0.3	1.60933 62546	5.9	−5.0
0.4	1.82279 32298	9.4	−11.0
0.5	2.04630 25124	13.1	−22.5
0.6	2.28413 68531	14.4	−44.7
0.7	2.54228 84689	+5.0	−88.4
0.8	2.82963 87346	−38.8	−177.6
0.9	3.16015 85865	−191.1	−369.0
1.0	3.55740 85377	−699.9	−813.0

SECTION 19.2. Multistep Methods, page 952

Purpose. To explain the idea of a multistep method in terms of the practically important Adams–Moulton method, a predictor–corrector method that in each computation uses four preceding values.

Main Content, Important Concepts

Adams–Bashforth method (5)

Adams–Moulton method (7)

Short Courses. This section may be omitted.

SOLUTIONS TO PROBLEM SET 19.2, page 955

2.

n	x_n	Starting y_n	Predicted y_n^*	Corrected y_n	Exact
0	0.0	1.000 000			
1	0.1	1.105 171			
2	0.2	1.221 403			
3	0.3	1.349 859			
4	0.4		1.491 821	1.491 825	1.491 825
5	0.5		1.648 717	1.648 722	1.648 721
6	0.6		1.822 114	1.822 120	1.822 119
7	0.7		2.013 748	2.013 754	2.013 753
8	0.8		2.225 536	2.225 543	2.225 541
9	0.9		2.459 598	2.459 605	2.459 603
10	1.0		2.718 277	2.718 285	2.718 282

4.

n	x_n	y_n	Exact	Error $\times 10^6$
0	1.0	0	0	0
1	1.1	0.104394	0.104394	0
2	1.2	0.215563	0.215563	0
3	1.3	0.331199	0.331199	0
4	1.4	0.449688	0.449886	−2.3
5	1.5	0.569871	0.569867	−3.5
6	1.6	0.690911	0.690907	−3.8
7	1.7	0.812198	0.812195	−3.9
8	1.8	0.933284	0.933280	−3.9

6. Solution $y^2 - x^2 = 8$.

x_n	y_n
1.2	3.07246
1.4	3.15595
1.6	3.24962
1.8	3.35261
2.0	3.46410
2.2	3.58330
2.4	3.70945
2.6	3.84188
2.8	3.97995
3.0	4.12311

8. $y = \tan x + x + 1$. $\tan x$ approaches infinity as $x \to \frac{1}{2}\pi$.

12. $y = e^{x^2}$. Some of the values and errors are:

x_n	y_n $(h = 0.05)$	Error $\times 10^6$	y_n $(h = 0.1)$	Error $\times 10^6$
0.1	1.010050		1.01005	
0.2	1.040817	−6	1.040811	
0.3	1.094188	−14	1.094224	−50
0.4	1.173535	−24	1.173623	−112
0.5	1.284064	−38	1.284219	−194
0.6	1.433388	−58	1.433636	−307
0.7	1.632404	−87	1.632782	−466
0.8	1.896612	−131	1.897175	−694
0.9	2.248105	−197	2.248931	−1023
1.0	2.718579	−297	2.719785	−1503

The errors differ by a factor 4 to 5, approximately.

14. $y_1 = 4.002707$, $y_2 = 4.022789$, $y_3 = 4.084511$, $y_4 = 4.230685$, $y_5 = 4.559046$, $y_6 = 5.364224$, $y_7 = 8.060954$. Exact: $y = \tan x - x + 4$

SECTION 19.3. Methods for Systems and Higher Order Differential Equations, page 956

Purpose. Extension of the methods in Sec. 19.1 to first-order systems and to higher order equations.

Content

Euler's method for systems (5)

Classical Runge–Kutta method extended to systems (6)

Runge–Kutta–Nyström method (7)

SOLUTIONS TO PROBLEM SET 19.3, page 961

2. See Fig. 82 in Sec. 3.3. The discussion in Sec. 3.3 is not needed for the present purpose. The computation gives:

x	y_1	y_2
0	0	4
0.2	0.8	3.2
0.4	1.28	2.4
0.6	1.504	1.664
0.8	1.536	1.0304
1.0	1.43488	0.51712

4. $y = 0, -0.15, -0.3, -0.44925, -0.596996, -0.742481$. The error increases monotone from 0 to 0.0081.

6. Much more accurate values.

x	$y(x)$	$10^8 \times$ Error of $y(x)$	$y'(x)$
0.1	0.804837	−8	−1.90484
0.2	0.618731	−15	−1.81873
0.3	0.440818	−20	−1.74082

8. We had to choose $x_0 \neq 0$ because of the factor $1/x$. Those initial values were taken from Ref. [1] in Appendix 1.

x	$J_0(x)$	$J_0'(x)$	$10^6 \times$ Error of $J_0(x)$
1	0.765198	−0.440051	0
1.5	0.511903	−0.558002	−76
2	0.224008	−0.576897	−117
2.5	−0.048289	−0.497386	−95
3	−0.260055	−0.339446	+3
3.5	−0.380298	−0.137795	170

10.

x_n	y_n	y_n'	y_n''	Exact (4S)	Error
0	0	0	1	0	0
0.2	0.02	0.2	1.21	0.0214	0.0014
0.4	0.0842	0.4420	1.4631	0.0918	0.0076
0.6	0.2019	0.7346	1.7682	0.2221	0.0202
0.8	0.3842	1.0883	2.1362	0.4255	0.0413
1.0	0.6446			0.7183	0.0737

12. $\Gamma(2/3) = (3/2)\Gamma(5/3)$ by (25); now use interpolation in Table A2, etc.

14.

x_n	y_n	k_1	k_2	k_3	k_4	$10^6 \times$ Error of y_n
1.0	0.765 198	−0.081 287	−0.056 989	−0.061 848	−0.034 604	0
1.5	0.511 819	−0.034 970	−0.007 296	−0.011 250	+0.015 740	+9
2.0	+0.223 946	+0.016 098	+0.041 840	+0.038 979	0.061 098	−55
2.5	−0.048 241	0.061 767	0.080 770	0.079 042	0.092 562	−143
3.0	−0.259 845	0.093 218	0.102 154	0.101 466	0.104 389	−207
3.5	−0.379 914					−214

In the present case the errors of the two methods are of the same order of magnitude. An exact comparison is not possible since the errors change sign in a different fashion in each method.

SECTION 19.4. Methods for Elliptic Partial Differential Equations, page 962

Purpose. To explain numerical methods for the Dirichlet problem involving the Laplace equation, the typical representative of elliptic equations.

Main Content, Important Concepts

Elliptic, parabolic, hyperbolic equations

Dirichlet, Neumann, mixed problems

Difference analogs (7), (8) of Poisson's and Laplace's equations

Coefficient scheme (9)

Liebmann's method of solution (identical with Gauss–Seidel, Sec. 18.3)

Peaceman–Rachford's ADI method (15)

Short Courses. Omit the ADI method.

Comments on Content

Neumann's problem and the mixed problem follow in the next section, including the modification in the case of irregular boundaries.

The distinction between the three kinds of equations (elliptic, parabolic, hyperbolic) is not merely a formal matter because the solutions of the three types behave differently in principle, and the boundary and initial conditions are different; this necessitates different numerical methods, as we shall see.

SOLUTIONS TO PROBLEM SET 19.4, page 969

2. 6 steps. Some results are

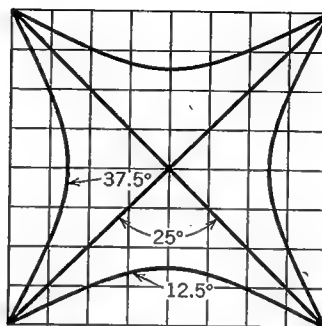
[93.75 90.625 65.625 64.0625]	(Step 2)
[87.8906 87.6953 62.6953 62.5977]	(Step 4)
[87.5244 87.5122 62.5122 62.5061]	(Step 6)
[87.5001 87.5 62.5 62.5]	(Step 10)

4. The values obtained by the Gauss elimination agree with those of the exact solution of the problem, $u(x, y) = x^3 - 3xy^2$. Gauss–Seidel would need 14 steps to produce 6S-values or 9 steps for 3S-values.

6. This shows the importance of good starting values; it then does not take long until the approximations come close to the solution. A rule of thumb is to take a rough estimate of the average of the boundary values at the points that enter the linear system. By starting from 0 we obtain

$$[0.094722 \quad 0.101487 \quad 0.317994 \quad 0.321376] \qquad \text{(Step 3)}$$
$$[0.107407 \quad 0.107830 \quad 0.324337 \quad 0.324548] \qquad \text{(Step 5).}$$

8. $u_{11} = 92.86$, $u_{21} = 90.18$, $u_{12} = 81.25$, $u_{22} = 75.00$, $u_{13} = 57.14$, $u_{23} = 47.32$, $u_{31} = u_{11}$, etc., by symmetry.

10. All the isotherms must begin and end at a corner. The diagonals are isotherms $u = 25$, because of the data obtained and for reasons of symmetry. Hence we obtain a qualitative picture as follows.

Section 19.4. Problem 10

12. (a) $u_{11} = -u_{12} = -66$

(b) By symmetry, we can reduce the problem to four equations in four unknowns. *Solution:*

$$\begin{aligned}
u_{11} = u_{31} &= -u_{15} = -u_{35} = -92.92\\
u_{21} &= -u_{25} = -87.45\\
u_{12} = u_{32} &= -u_{14} = -u_{34} = -64.22\\
u_{22} &= -u_{24} = -53.98\\
u_{13} = u_{23} &= u_{33} = 0
\end{aligned}$$

14. ***First step.*** First come ***rows*** $j = 1, j = 2$; for these, (14a) is

$$\begin{aligned}
j = 1,\ i = 1.\quad & u_{01} - 4u_{11} + u_{21} = -u_{10} - u_{12}\\
i = 2.\quad & u_{11} - 4u_{21} + u_{31} = -u_{20} - u_{22}\\
j = 2,\ i = 1.\quad & u_{02} - 4u_{12} + u_{22} = -u_{11} - u_{13}\\
i = 2.\quad & u_{12} - 4u_{22} + u_{32} = -u_{21} - u_{23}.
\end{aligned}$$

Six of the boundary values are zero, and the two on the upper edge are $u_{13} = u_{23} = \sqrt{3}/2 = 0.866\,025$. Also, on the right we substitute the starting values 0. With this,

our four equations become

$$\begin{aligned} -4u_{11} + \ u_{21} &= 0 \\ u_{11} - 4u_{21} &= 0 \\ -4u_{12} + \ u_{22} &= -0.866\,025 \\ u_{12} - 4u_{22} &= -0.866\,025. \end{aligned}$$

The solution is from the first two equations

$$u_{11} = 0, \qquad u_{21} = 0$$

and from the other two equations

$$u_{12} = 0.288\,675, \qquad u_{22} = 0.288\,675.$$

First step. Now come ***columns;*** for these, (14b) is

$$\begin{aligned} i = 1, \ j = 1. \quad & u_{10} - 4u_{11} + u_{12} = -u_{01} - u_{21} \\ j = 2. \quad & u_{11} - 4u_{12} + u_{13} = -u_{02} - u_{22} \\ i = 2, \ j = 1. \quad & u_{20} - 4u_{21} + u_{22} = -u_{11} - u_{31} \\ j = 2. \quad & u_{21} - 4u_{22} + u_{23} = -u_{12} - u_{32}. \end{aligned}$$

With the boundary values and the previous solution ***on the right,*** this becomes

$$\begin{aligned} -4u_{11} + \ u_{12} &= 0 \\ u_{11} - 4u_{12} &= -0.866\,025 - 0.288\,675 \\ -4u_{21} + \ u_{22} &= 0 \\ u_{21} - 4u_{22} &= -0.866\,025 - 0.288\,675. \end{aligned}$$

The solution is

$$\begin{aligned} u_{11} &= 0.076\,98 \\ u_{21} &= 0.076\,98 \\ u_{12} &= 0.307\,92 \\ u_{22} &= 0.307\,92. \end{aligned}$$

Second step. Rows. We can use the previous equations, changing only the right sides:

$$\begin{aligned} -4u_{11} + \ u_{21} &= -0.307\,92 \\ u_{11} - 4u_{21} &= -0.307\,92 \\ -4u_{12} + \ u_{22} &= -0.866\,025 - 0.076\,98 \\ u_{12} - 4u_{22} &= -0.866\,025 - 0.076\,98 \end{aligned}$$

Solution:

$$u_{11} = u_{21} = 0.102\,640, \qquad u_{12} = u_{22} = 0.314\,335.$$

Second step. Columns. The equations with the new right sides are

$$\begin{aligned} -4u_{11} + \ u_{12} &= -0.102\,640 \\ u_{11} - 4u_{12} &= -0.866\,025 - 0.314\,335 \\ -4u_{21} + \ u_{22} &= -0.102\,640 \\ u_{21} - 4u_{22} &= -0.866\,025 - 0.314\,335. \end{aligned}$$

Final result (solution of these equations):

$$u_{11} = 0.106\,061$$
$$u_{21} = 0.106\,061$$
$$u_{12} = 0.321\,605$$
$$u_{22} = 0.321\,605.$$

Exact 3D values:

$$u_{11} = u_{21} = 0.108, \qquad u_{12} = u_{22} = 0.325.$$

16. CAS PROJECT. (b) The solution of the linear system (rounded to integers), with the values arranged as the points in the xy-plane, is

160	170	157	110
138	145	125	75
138	145	125	75
160	170	157	110

Twenty steps gave accuracies of 3S–5S, with slight variations between the components of the output vector.

SECTION 19.5. Neumann and Mixed Problems. Irregular Boundary, page 971

Purpose. Continuing our discussion of elliptic equations, we explain the ideas needed for handling Neumann and mixed problems and the modifications required when the domain is no longer a rectangle.

Main Content, Important Concepts

Mixed problem for a Poisson equation (Example 1)

Modified stencil (6) (notation in Fig. 428)

Comments on Content

Neumann's problem can be handled as explained in Example 1.

In all the cases of an elliptic equation we need only one boundary condition at each point (given u or given u_n).

SOLUTIONS TO PROBLEM SET 19.5, page 975

2. $0 = u_{01,x} = \dfrac{1}{2h}(u_{11} - u_{-1,1})$ gives $u_{-1,1} = u_{11}$. Similarly, $u_{41} = u_{21} + 3$ from the condition on the right edge, so that the equations are

$$\begin{aligned}
-4u_{01} + 2u_{11} \qquad\qquad &= 1 \\
u_{01} - 4u_{11} + u_{21} \qquad &= -0.25 + 0.75 = 0.5 \\
u_{11} - 4u_{21} + u_{31} &= -1 \\
2u_{21} - 4u_{31} &= -2.25 - 1.25 - 3 = -6.5.
\end{aligned}$$

$u_{01} = -0.25$, $u_{11} = 0$, $u_{21} = 0.75$, $u_{31} = 2$; this agrees with the values of the exact solution $u(x, y) = x^2 - y^2$ of the problem.

4. The exact solution of the Poisson equation is $u = x^2y^2$. The approximate solution results from $\mathbf{Au} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 2 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 2 & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 10 \\ 8 \\ 1 \\ -20 \\ -103 \end{bmatrix}$$

where the six equations correspond to $P_{11}, P_{21}, P_{31}, P_{12}, P_{22}, P_{32}$, in our usual order. The components of $\mathbf{b}$ are of the form $a - c$ with a resulting from $2(x^2 + y^2)$ and c from the boundary values; thus, $4 - 0 = 4$, $10 - 0 = 10$, $20 - 12 = 8$, $10 - 9 = 1$, $16 - 36 = -20$, $26 - 81 - 48 = -103$. The solution of this system agrees with the values obtained at the P_{jk} from the exact solution, $u_{11} = 1$, $u_{21} = u_{12} = 4$, $u_{22} = 16$, and $u_{31} = 9$, $u_{32} = 36$ on the boundary. $u_{41} = u_{21} + 12$ and $u_{42} = u_{22} + 48$ produced entries 2 in $\mathbf{A}$ and -12 and -48 in $\mathbf{b}$.

6. Exact solution $u = 9y \sin \frac{1}{3}\pi x$. Linear system $\mathbf{Au} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 2 & 0 & -4 & 1 \\ 0 & 0 & 0 & 2 & 1 & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} a \\ a \\ 2a \\ 2a \\ 3a + c \\ 3a + c \end{bmatrix}$$

$a = -8.54733$, $c = -\sqrt{243} = -15.5885$. The solution of this system is (exact values of u in parentheses)

$$u_{11} = u_{21} = 8.46365 \text{ (exact } \tfrac{9}{2}\sqrt{3} = 7.79423)$$
$$u_{12} = u_{22} = 16.8436 \text{ (exact } 9\sqrt{3} = 15.5885)$$
$$u_{13} = u_{23} = 24.9726 \text{ (exact } \tfrac{27}{2}\sqrt{3} = 23.3827).$$

14. Let v denote the unknown boundary potential. Then v occurs in $\mathbf{Au} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & \frac{2}{3} & \frac{2}{3} & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ -v \\ -v \\ -\frac{8}{3}v \end{bmatrix}.$$

The solution of this linear system is $\mathbf{u} = \dfrac{v}{19}[5 \quad 10 \quad 10 \quad 16]^{\mathsf{T}}$. From this and $5v/19 = 100$ (the potential at P_{11}) we have $v = 380$ as the constant boundary potential on the indicated portion of the boundary.

16. Two equations are as usual:

$$\begin{aligned} -4u_{11} + u_{21} + u_{12} - 2 &= 2 \\ u_{11} - 4u_{21} \qquad\quad - 0.5 &= 2 \end{aligned}$$

where the right side is due to the fact that we are dealing with the Poisson equation. The third equation results from (6) with $a = p = q = 1$ and $b = 1/2$. We get

$$2\left[\frac{u_{22}}{2} + \frac{u_{1,5/2}}{3/4} + \frac{u_{02}}{2} + \frac{u_{11}}{3/2} - \frac{3/2}{1/2}u_{12}\right] = 2.$$

The first two terms are zero and $u_{02} = -2$; these are given boundary values. There remains

$$\tfrac{4}{3}u_{11} - 6u_{12} = 4.$$

Our three equations for the three unknowns have the solution

$$u_{11} = -1.5, \quad u_{21} = -1, \quad u_{12} = -1.$$

SECTION 19.6. Methods for Parabolic Equations, page 976

Purpose. To show the numerical solution of the heat equation, the prototype of a parabolic equation, on the region given by $0 \leqq x \leqq 1$, $t \geqq 0$, subject to one initial condition (initial temperature) and one boundary condition on each of the two vertical boundaries.

Content

Direct method based on (5), convergence condition (6)

Crank–Nicolson method based on (8)

Special case (9) of (8)

Comment on Content

Condition (6) restricts the size of time steps too much, a disadvantage that the Crank–Nicolson method avoids.

SOLUTIONS TO PROBLEM SET 19.6, page 981

4. The first term in (10), Sec. 11.5, gives $\exp\left[-\frac{1}{100}\pi^2 t\right] = 0.1$, $t = 100\,(\ln 10)/\pi^2 = 23.3$. The other terms decrease much more rapidly and contribute practically nothing.

6. $u(x, 0) = u(1 - x, 0)$ and the boundary conditions imply $u(x, t) = u(1 - x, t)$ for all t. The calculation gives

(0, 0.2, 0.35, 0.35, 0.2, 0)

(0, 0.1875, 0.3125, 0.3125, 0.1875, 0)

(0, 0.171875, 0.28125, 0.28125, 0.171875, 0)

(0, 0.15625, 0.253906, 0.253906, 0.15625, 0)

(0, 0.141602, 0.229492, 0.229492, 0.141602, 0)

8. We have $k = 0.01$. The boundary condition on the left is that the normal derivative is zero. Now if we were at an inner point, we would have, by (5),

$$u_{0,j+1} = \tfrac{1}{4}u_{-1,j} + \tfrac{1}{2}u_{0j} + \tfrac{1}{4}u_{1j}.$$

Here, by the central difference formula for the normal derivative (partial derivative with respect to x) we get

$$0 = \frac{\partial u_{0j}}{\partial x} = \frac{1}{2h}(u_{1j} - u_{-1,j})$$

so that the previous formula gives what we need,

$$u_{0,j+1} = \tfrac{1}{2}(u_{0j} + u_{1j}).$$

The underlying idea is quite similar to that in Sec. 19.5. The computation gives

t	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1$
0	0	0	0	0	0	0
0.01	0	0	0	0	0	0.5
0.02	0	0	0	0	0.125	0.866 025
0.03	0	0	0	0.031	0.279	1
0.04	0	0	0.008	0.085	0.397	0.866 025
0.05	0	0.002	0.025	0.144	0.437	0.5
0.06	0.001	0.007	0.049	0.187	0.379	0
0.07	0.004	0.016	0.073	0.201	0.236	−0.5
0.08	0.010	0.027	0.091	0.178	0.043	−0.866 025
0.09	0.019	0.039	0.097	0.122	−0.601	−1
0.10	0.029	0.048	0.089	−0.065	−0.520	−0.866 025
0.11	0.039	0.054	0.040	−0.140	−0.493	−0.5
0.12	0.046	0.047	−0.002	−0.183	−0.406	0

10. $r = k/h^2 = 1$, $k = 1$, 2 steps. The series in Sec. 11.5 gives (with $L = 10$, $t = 2$, $b_1 = 2.58012$, $b_2 = 0$, $b_3 = 0.09556$)

$$u = b_1 \sin 0.1x \exp(-\pi^2/50) + b_3 \sin 0.3x \exp(-9\pi^2/50).$$

The values for $t = 2$ and $x = 0, 1, \cdots, 10$ are (exact values in parentheses) 0 (0), 0.6691 (0.6546), 1.2619 (1.2449), 1.7212 (1.7135), 2.0075 (2.0143), 2.1043 (2.1179), 2.0075 (2.0143), etc. (symmetric).

12. CAS PROJECT. $u(0, t) = u(1, t) = 0$, $u(0.2, t) = u(0.8, t)$, $u(0.4, t) = u(0.6, t)$, where

	$x = 0.2$	$x = 0.4$	
$t = 0$	0.587785	0.951057	
$t = 0.04$	0.393432	0.636586	Explicit
	0.399274	0.646039	CN
	0.396065	0.640846	Exact (6D)
$t = 0.08$	0.263342	0.426096	
	0.271221	0.438844	
	0.266878	0.431818	
$t = 0.12$	0.176267	0.285206	
	0.184236	0.298100	
	0.179829	0.290970	
$t = 0.16$	0.117983	0.190901	
	0.125149	0.202495	
	0.121174	0.196063	
$t = 0.2$	0.078972	0.127779	
	0.085012	0.137552	
	0.081650	0.132112	

SECTION 19.7. Methods for Hyperbolic Equations, page 982

Purpose. Explanation of the numerical solution of the wave equation, the prototype of a hyperbolic equation, on a region of the same type as in the last section, subject to initial and boundary conditions that guarantee the uniqueness of the solution.

Comments on Content

We now have two initial conditions (given initial displacement and given initial velocity), in contrast to the heat equation in the last section, where we had only one initial condition.

The computation by (6) is simple. Formula (8) gives the values of the first time steps in terms of the initial data.

SOLUTIONS TO PROBLEM SET 19.7, page 984

2. Note that the curve of $f(x)$ is no longer symmetric with respect to $x = 0.5$. The solution was required for $0 \leqq t \leqq 1$. We present it here for a full cycle $0 \leqq t \leqq 2$:

t	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$
0	0.032	0.096	0.144	0.128
0.2	0.048	0.088	0.112	0.072
0.4	0.056	0.064	0.016	−0.016
0.6	0.016	−0.016	−0.064	−0.056
0.8	−0.072	−0.112	−0.088	−0.048
1.0	−0.128	−0.144	−0.096	−0.032
1.2	−0.072	−0.112	−0.088	−0.048
1.4	0.016	−0.016	−0.064	−0.056
1.6	0.056	0.064	0.016	−0.016
1.8	0.048	0.088	0.112	0.072
2.0	0.032	0.096	0.144	0.128

4. By (14), Sec. 11.4, with $c = 1$ the left side of (6) is

(A) $$u_{i,j+1} = u(ih, (j + 1)h) = \tfrac{1}{2}[f(ih + (j + 1)h) + f(ih - (j + 1)h)]$$

and the right side is the sum of the six terms

$$u_{i-1,j} = \tfrac{1}{2}[f((i - 1)h + jh) + f((i - 1)h - jh)],$$
$$u_{i+1,j} = \tfrac{1}{2}[f(i + 1)h + jh) + f((i + 1)h - jh)],$$
$$-u_{i,j-1} = -\tfrac{1}{2}[f(ih + (j - 1)h) + f(ih - (j - 1)h].$$

Four of these six terms cancel in pairs, and the remaining expression equals the right side of (A).

6. From (13), Sec. 11.4, with $c = 1$ we get the exact solution

$$u(x, t) = \frac{1}{2}\int_{x-ct}^{x+ct} \sin \pi s \, ds = \frac{1}{2\pi}[\cos \pi(x - ct) - \cos \pi(x + ct)].$$

From (8) we have $kg_i = 0.1g_i = 0.1 \sin 0.1\pi i$. Because of the symmetry with respect

to $x = 0.5$ we may list only the following values (with the exact values in parentheses):

t	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
0.0	0	0	0	0	0
0.1	0.030 902 (0.030 396)	0.058 779 (0.057 816)	0.080 902 (0.079 577)	0.095 106 (0.093 549)	0.100 000 (0.098 363)
0.2	0.058 779 (0.057 816)	0.111 804 (0.109 973)	0.153 885 (0.151 365)	0.180 902 (0.177 941)	0.190 212 (0.187 098)
0.3	0.080 902 (0.079 577)	0.153 885 (0.151 365)	0.211 804 (0.208 337)	0.248 991 (0.244 914)	0.261 804 (0.257 518)
0.4	0.095 106 (0.093 549)	0.180 902 (0.177 941)	0.248 991 (0.244 914)	0.292 706 (0.287 914)	0.307 770 (0.302 731)

8. Since $u(x, 0) = f(x)$, the derivation is immediate. Formula (8) results if the integral equals $2kg_i$.

10. Exact solution: $u(x, t) = (x + t)^2$. The values obtained in the computation are those of the exact solution. $u_{11}, u_{21}, u_{31}, u_{41}$ are obtained from (8) and the initial conditions $u_{i0} = (0.2i)^2$, $g_i = 0.2i$. In connection with the left boundary condition we can use the central difference formula

$$\frac{1}{2h}(u_{1,j} - u_{-1,j}) \approx u_x(0, jk) = 2jk$$

to obtain $u_{-1,j}$ and then (8) to compute u_{01} and (6) to compute $u_{0,j+1}$.

SOLUTIONS TO CHAPTER 19 REVIEW, page 984

22. $y = e^x$. Computed values are

x_n	y_n	$y(x_n)$	Error $\times 10^6$	Error in Prob. 21
0.01	1.010 000	1.010 050	50	
0.02	1.020 100	1.020 201	101	
0.03	1.030 301	1.030 455	154	
0.04	1.040 604	1.040 811	207	
0.05	1.051 010	1.051 271	261	
0.06	1.061 520	1.061 837	316	
0.07	1.072 135	1.072 508	373	
0.08	1.082 857	1.083 287	430	
0.09	1.093 685	1.094 174	489	
0.10	1.104 622	1.105 171	549	0.005 171

We see that the error of the last value has decreased by a factor 10, approximately, due to the smaller step.

24. $y = 2e^{-x} + x^2 + 1.$

x_n	y_n	Error $\times 10^4$
0.1	2.8205	−8
0.2	2.6790	−15
0.3	2.5738	−22
0.4	2.5033	−27
0.5	2.4662	−32
0.6	2.4612	−36
0.7	2.4871	−39
0.8	2.5429	−42
0.9	2.6276	−44
1.0	2.7404	−46

26. $y = e^{x^2/2}$; $0, 7 \cdot 10^{-9}, 7 \cdot 10^{-8}, 4 \cdot 10^{-7}, 10^{-6}, 5 \cdot 10^{-6}$

28. From $y' = x + y$ and the given formula we get, with $h = 0.2$,

$$\begin{aligned} k_1 &= 0.2(x_n + y_n) \\ k_2 &= 0.2[x_n + 0.1 + y_n + 0.1(x_n + y_n)] \\ &= 0.2[1.1(x_n + y_n) + 0.1] \\ k_3^* &= 0.2[x_n + 0.2 + y_n - 0.2(x_n + y_n) + 0.4[1.1(x_n + y_n) + 0.1]] \\ &= 0.2[1.24(x_n + y_n) + 0.24] \end{aligned}$$

and from this

$$y_{n+1} = y_n + \tfrac{1}{6}[1.328(x_n + y_n) + 0.128].$$

The computed values are

x_n	y_n	Error $\times 10^6$
0.0	0.000 000	0
0.2	0.021 333	69
0.4	0.091 655	170
0.6	0.221 808	311
0.8	0.425 035	506
1.0	0.717 509	772

30. Solution $y = \tan x - x + 4$.

x_n	y_n	Error $\times 10^6$
0.8	4.22969	−52
1.0	4.55686	+548

The starting values were obtained by classical Runge–Kutta.

34. $y_1 = 2, 2, 1.68, \cdots, -3.30955$; $y_2 = 0, -1.6, -3.2, \cdots, 5.17403$. Exact solution $4y_1^2 + y_2^2 = 16$ (ellipse).

36. $y_1 = -6e^{9x} + 3e^{3x}$, $y_2 = -2e^{9x} - e^{3x}$; errors of y_1: $-1 \times 10^{-3}, -3 \times 10^{-3}, -7 \times 10^{-3}$; errors of y_2: $-3 \times 10^{-4}, -10 \times 10^{-4}, -25 \times 10^{-4}$

38. The computed values are

x_n	y_n	y'_n	y_{exact}	Error	y'_{exact}	Error
0.0	0	−3	0	0	−3	0
0.1	−0.3	−3	−0.299	0.001	−2.97	0.03
0.2	−0.597	−2.94	−0.592	0.005	−2.88	0.06
0.3	−0.884 985	−2.819 700	−0.873	0.011 985	−2.73	0.089 7
0.4	−1.157 910	−2.638 796	−1.136	0.021 910	−2.52	0.118 796
0.5	−1.409 698		−1.375	0.034 698	−2.25	

40. $u(P_{11}) = u(P_{12}) = 105$, $u(P_{21}) = 155$, $u(P_{22}) = 115$

42. 1.96, 7.86, 29.46

44. From the 3D-values given below we see that at each point $x > 0$ the temperature oscillates with a phase lag and a maximum amplitude that decreases with decreasing x.

t	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
0	0	0	0	0	0	0
0.02	0	0	0	0	0	0.5
0.04	0	0	0	0	0.250	0.866 025
0.06	0	0	0	0.125	0.433	1
0.08	0	0	0.062	0.217	0.562	0.866 025
0.10	0	0.031	0.108	0.312	0.541	0.5
0.12	0	0.054	0.172	0.325	0.406	0
0.14	0	0.086	0.189	0.289	0.162	−0.5
0.16	0	0.095	0.188	0.176	−0.105	−0.866 025
0.18	0	0.094	0.135	0.041	−0.345	−1
0.20	0	0.068	0.067	−0.105	−0.479	−0.866 025
0.22	0	0.034	−0.019	−0.206	−0.485	−0.5
0.24	0	−0.009	−0.086	−0.252	−0.353	0

PART F. OPTIMIZATION. GRAPHS

CHAPTER 20 Unconstrained Optimization. Linear Programming

Major Change

The simplex method of linear programming has been completely rewritten in the spirit of matrix techniques, without making reference to other chapters (6 or 18).

SECTION 20.1. Basic Concepts. Unconstrained Optimization, page 990

Purpose. To explain the concepts needed throughout this chapter. To discuss Cauchy's method of steepest descent or gradient method, a popular method of unconstrained optimization.

Main Content, Important Concepts

Objective function

Control variables

Constraints, unconstrained optimization

Cauchy's method

SOLUTIONS TO PROBLEM SET 20.1, page 993

2. $f(\mathbf{x}) = (x_1 - 3)^2 + 4(x_2 - 1)^2 - 13$. Calculation gives

x_1	x_2
3.73846	1.04615
3.11077	0.889231
3.0818	1.00511

4. $f(\mathbf{x}) = 0.8(x_1 + 1.4)^2 + 0.35(x_2 + 0.3)^2 + const.$ Steps 1–3 give

x_1	x_2
−1.48804	0.979908
−1.29286	−0.261496
−1.40147	−0.278573

6. $f(\mathbf{x}) = x_1{}^2 - x_2$ gives

$$\mathbf{z}(t) = \mathbf{x} - t[2x_1, \quad -1] = [(1 - 2t)x_1, \quad x_2 + t],$$

hence

$$g(t) = (1 - 2t)^2 x_1{}^2 - x_2 - t,$$
$$g'(t) = -4(1 - 2t)x_1{}^2 - 1 = 0.$$

From this,

$$1 - 2t = -\frac{1}{4x_1^2}, \qquad t = \frac{1}{2} + \frac{1}{8x_1^2}.$$

For this t,

$$\mathbf{z}(t) = \left[-\frac{1}{4x_1}, \quad x_2 + \frac{1}{2} + \frac{1}{8x_1^2} \right].$$

From this, with $x_1 = 1$, $x_2 = 1$, we get successively

$$\mathbf{z}_{(1)} = \left[-\tfrac{1}{4}, \quad 1 + \tfrac{1}{2} + \tfrac{1}{8}\right],$$
$$\mathbf{z}_{(2)} = \left[1, \quad 1 + 2 \cdot \tfrac{1}{2} + \tfrac{1}{8} + 2\right],$$
$$\mathbf{z}_{(3)} = \left[-\tfrac{1}{4}, \quad 1 + 3 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{8} + 2\right], \quad \text{etc.}$$

The student should sketch this, to see that it is reasonable. The process continues indefinitely, as had to be expected.

8. The calculation gives for steps 1–5

x_1	x_2
−1.33333	2.66667
−3.55556	−1.77778
2.37037	−4.74074
6.32099	3.16049
−4.21399	8.42798

This is the beginning of a broken line of segments spiraling away from the origin. At the corner points, f is alternatingly positive and negative and increases monotone in absolute value.

10. CAS PROJECT. (c) For $f(\mathbf{x}) = x_1^2 + x_2^4$ the values converge relatively rapidly to $[0 \quad 0]^T$:

x_1	x_2
0.410245	−0.589755
−0.00977922	−0.16973
0.007137	−0.152814
−0.00550786	−0.140169
0.00441861	−0.130242
−0.00364745	−0.122176

Similarly for $f(\mathbf{x}) = x_1^4 + x_2^4$:

x_1	x_2
−0.352941	0.705882
−0.249135	−0.124567
0.043965	−0.08793
0.0310341	0.0155171
−0.00547661	0.0109532
−0.00386584	−0.00193292

SECTION 20.2. Linear Programming, page 994

Purpose. To discuss the basic ideas of linear programming in terms of very simple examples.

Main Content, Important Concepts

Linear programming problem

Its normal form. Slack variables

Feasible solution, basic feasible solution

Optimal solution

Comments on Content

Whereas the function to be maximized (or minimized) by Cauchy's method was arbitrary (differentiable), but we had no constraints, we now simply have a linear objective function, but constraints, so that calculus no longer helps.

No systematic method of solution is discussed in this section; these follow in the next sections.

SOLUTIONS TO PROBLEM SET 20.2, page 997

2. No. For instance, $f = 5x_1 + 2x_2$ yields maximum profit $f = 12$ for every point on the segment AB.

6. Ordinarily a vertex of a region is the intersection of only *two* straight lines given by inequalities taken with the equality sign. Here, (5, 4) is the intersection of *three* such lines. This may merit special attention in some cases, as we discuss in Sec. 20.4.

8. The first inequality could be dropped from the problem because it does not restrict the region determined by the other inequalities. Note that that region is unbounded (stretches to infinity). This would cause a problem in maximizing an objective function with positive coefficients.

10. $f(9, 4) = 270 + 40 = 310$ is the maximum.

12. No solution because the region is unbounded

14. $f_{\max} = f(9, 6) = 360$

16. $f = x_1 + x_2$, $2x_1 + 4x_2 \leqq 800$, $5x_1 + 2x_2 \leqq 600$, $x_1 = 50$, $x_2 = 175$, $f_{\max} = f(50, 175) = 225$

18. x_1 = Number of days of operation of kiln I, x_2 = Number of days of operation of kiln II. Objective function $f = 400x_1 + 600x_2$. Constraints:

$$\begin{aligned} 3000x_1 + 2000x_2 &\geqq 9000 && \text{(Grey bricks)} \\ 2000x_1 + 5000x_2 &\geqq 17000 && \text{(Red bricks)} \\ 300x_1 + 1500x_2 &\geqq 4500 && \text{(Glazed bricks).} \end{aligned}$$

$f_{\min} = f(1, 3) = 2200$, as can be seen from a sketch of the region in the x_1x_2-plane resulting from the constraints in the first quadrant. Operate kiln I one day and kiln II three days in filling that order. Note that the region determined by the constraints in the first quadrant of the x_1x_2-plane is unbounded, which causes no difficulty because we minimize (not maximize) the objective function.

20. x_1 units of A and x_2 units of B cost $f = 1.5x_1 + 2x_2$. Constraints are

$$10x_1 + 35x_2 \geqq 100 \quad \text{(Protein)}$$
$$700x_1 + 500x_2 \geqq 3100 \quad \text{(Calories).}$$

From a sketch of the region we see that $f_{\min} = f(3, 2) = 8.50$. Hence the minimum cost diet consists of 3 units A and 2 units B.

SECTION 20.3. Simplex Method, page 998

Purpose. To discuss the standard method of linear programming for systematically finding an optimal solution by a finite sequence of transformations of matrices.

Main Content, Important Concepts

Normal form of the problem

Initial simplex table

Pivoting, further simplex tables (augmented matrices)

Comment on Concepts and Method

The given form of the problem involves inequalities. By introducing slack variables we convert the problem to the normal form. This is a linear system of equations. The initial simplex table is its augmented matrix. It is transformed by first selecting the column of a pivot and then the row of that pivot. The rules for this are entirely different from those for pivoting in connection with the solution of a linear system of equations. The selection of a pivot is followed by a process of elimination by row operations similar to that in the Gauss–Jordan method (Sec. 6.7). This is the first step, leading to another simplex table (another augmented matrix). The next step is done by the same rules, and so on. The process comes to an end when the first row of the simplex table obtained contains no more negative entries. From this final simplex table one can read the optimal solution of the problem.

SOLUTIONS TO PROBLEM SET 20.3, page 1001

2. The normal form is

$$\begin{aligned} z - x_1 - x_2 \qquad\qquad &= 0 \\ 2x_1 + 4x_2 + x_3 \qquad &= 800 \\ 5x_1 + 2x_2 \qquad + x_4 &= 600. \end{aligned}$$

The calculation is

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 1 & 0 & 800 \\ 0 & 5 & 2 & 0 & 1 & 600 \end{bmatrix}$$

$800/2 = 400$, $600/5 = 120$, pivot 5

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -\frac{3}{5} & 0 & \frac{1}{5} & 120 \\ 0 & 0 & \frac{16}{5} & 1 & -\frac{2}{5} & 560 \\ 0 & 5 & 2 & 0 & 1 & 600 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{1}{5}\text{ Row 3} \\ \text{Row 2} - \frac{2}{5}\text{ Row 3} \\ \end{matrix}$$

$560/\frac{16}{5} < 600/2$, pivot 16/5

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{16} & \frac{1}{8} & 225 \\ 0 & 0 & \frac{16}{5} & 1 & -\frac{2}{5} & 560 \\ 0 & 5 & 0 & -\frac{5}{8} & \frac{5}{4} & 250 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{3}{16}\text{ Row 2} \\ \\ \text{Row 3} - \frac{5}{8}\text{ Row 2} \end{matrix}$$

$x_1 = 250/5 = 50$, $x_2 = 560/\frac{16}{5} = 175$, $z_{max} = f(50, 175) = 225$.

4. The matrices and pivot selections are

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 & 550 \\ 0 & 5 & 4 & 0 & 1 & 650 \end{bmatrix}$$

$550/3 > 650/5$, pivot 5

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -\frac{1}{5} & 0 & \frac{1}{5} & 130 \\ 0 & 0 & \frac{8}{5} & 1 & -\frac{3}{5} & 160 \\ 0 & 5 & 4 & 0 & 1 & 650 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{1}{5}\text{ Row 3} \\ \text{Row 2} - \frac{3}{5}\text{ Row 3} \\ \\ \end{matrix}$$

$160/\frac{8}{5} < 650/4$, pivot 8/5

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 150 \\ 0 & 0 & \frac{8}{5} & 1 & -\frac{3}{5} & 160 \\ 0 & 5 & 0 & -\frac{5}{2} & \frac{5}{2} & 250 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{1}{8}\text{ Row 2} \\ \\ \text{Row 3} - \frac{5}{2}\text{ Row 2} \end{matrix}$$

$f_{max} = 150$ at $x_1 = 250/5 = 50$, $x_2 = 160/(8/5) = 100$.

6. The matrices and pivot selections are

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -90 & -50 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 18 \\ 0 & 1 & 1 & 0 & 1 & 0 & 10 \\ 0 & 3 & 1 & 0 & 0 & 1 & 24 \end{bmatrix}$$

pivot 3 in row 4

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -20 & 0 & 0 & 30 & 720 \\ 0 & 0 & \frac{8}{3} & 1 & 0 & -\frac{1}{3} & 10 \\ 0 & 0 & \frac{2}{3} & 0 & 1 & -\frac{1}{3} & 2 \\ 0 & 3 & 1 & 0 & 0 & 1 & 24 \end{bmatrix}$$

pivot 2/3 in row 3

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 30 & 20 & 780 \\ 0 & 0 & 0 & 1 & -4 & 1 & 2 \\ 0 & 0 & \frac{2}{3} & 0 & 1 & -\frac{1}{3} & 2 \\ 0 & 3 & 0 & 0 & -\frac{3}{2} & \frac{3}{2} & 21 \end{bmatrix}$$

$f_{max} = 780$ at $x_1 = 21/3 = 7$, $x_2 = 2/\frac{2}{3} = 3$.

8. The matrices and pivot selections are

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -4 & 10 & 20 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 5 & 1 & 0 & 0 & 60 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 20 \\ 0 & 2 & 0 & 3 & 0 & 0 & 1 & 30 \end{bmatrix}$$

$60/4 = 15 < 20/1 = 20$, pivot 4

$$\mathbf{T}_1 = \begin{bmatrix} 1 & -\frac{23}{2} & 0 & \frac{15}{2} & -\frac{5}{2} & 0 & 0 & -150 \\ 0 & 3 & 4 & 5 & 1 & 0 & 0 & 60 \\ 0 & \frac{5}{4} & 0 & -\frac{5}{4} & -\frac{1}{4} & 1 & 0 & 5 \\ 0 & 2 & 0 & 3 & 0 & 0 & 1 & 30 \end{bmatrix}$$

$60/5 > 30/3$, pivot 3

$$\mathbf{T}_2 = \begin{bmatrix} 1 & -\frac{33}{2} & 0 & 0 & -\frac{5}{2} & 0 & -\frac{5}{2} & -225 \\ 0 & -\frac{1}{3} & 4 & 0 & 1 & 0 & -\frac{5}{3} & 10 \\ 0 & \frac{25}{12} & 0 & 0 & -\frac{1}{4} & 1 & \frac{5}{12} & \frac{35}{2} \\ 0 & 2 & 0 & 3 & 0 & 0 & 1 & 30 \end{bmatrix}$$

$f_{\min} = -225$ at $x_1 = 0$, $x_2 = 10/4 = 2.5$, $x_3 = 30/3 = 10$.

SECTION 20.4. Simplex Method: Degeneracy, Difficulties in Starting, page 1002

Purpose. To explain ways of overcoming difficulties that may arise in applying the simplex method.

Main Content, Important Concepts

Degenerate feasible solution

Artificial variable (for overcoming difficulties in starting)

SOLUTIONS TO PROBLEM SET 20.4, page 1007

2. In the second step in Prob. 1 we had a choice of the pivot, and in the present problem, due to our rule of choice, we took the other pivot. The result remained the same. The calculation is

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -6 & -12 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 6 & 12 & 0 & 0 & 1 & 72 \\ 0 & 0 & 1 & 0 & 1 & 0 & 4 \end{bmatrix}$$

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -12 & 6 & 0 & 0 & 24 \\ 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 12 & -6 & 0 & 1 & 48 \\ 0 & 0 & 1 & 0 & 1 & 0 & 4 \end{bmatrix} \quad \begin{matrix} \text{R1} + 6\ \text{R2} \\ \text{R2} \\ \text{R3} - 6\ \text{R2} \\ \text{R4} \end{matrix}$$

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 72 \\ 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 12 & -6 & 0 & 1 & 48 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{12} & 0 \end{bmatrix} \quad \begin{matrix} \text{R1 + R3} \\ \text{R2} \\ \text{R3} \\ \text{R4} - \frac{1}{12}\text{R3} \end{matrix}$$

This gives $x_1 = 4$, $x_2 = 48/12 = 4$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, $f(4, 4) = 72$.

4. The calculation is as follows.

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -300 & -500 & 0 & 0 & 0 & 0 \\ 0 & 2 & 8 & 1 & 0 & 0 & 60 \\ 0 & 2 & 1 & 0 & 1 & 0 & 30 \\ 0 & 4 & 4 & 0 & 0 & 1 & 60 \end{bmatrix}$$

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -350 & 0 & 150 & 0 & 4500 \\ 0 & 0 & 7 & 1 & -1 & 0 & 30 \\ 0 & 2 & 1 & 0 & 1 & 0 & 30 \\ 0 & 0 & 2 & 0 & -2 & 1 & 0 \end{bmatrix} \quad \begin{matrix} \text{R1 + 150 R3} \\ \text{R2 − R3} \\ \text{R3} \\ \text{R4 − 2 R3} \end{matrix}$$

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & -200 & 175 & 4500 \\ 0 & 0 & 0 & 1 & 6 & -\frac{7}{2} & 30 \\ 0 & 2 & 0 & 0 & 2 & -\frac{1}{2} & 30 \\ 0 & 0 & 2 & 0 & -2 & 1 & 0 \end{bmatrix} \quad \begin{matrix} \text{R1 + 175 R4} \\ \text{R2} - \frac{7}{2}\text{R4} \\ \text{R3} - \frac{1}{2}\text{R4} \\ \text{R4} \end{matrix}$$

$z = 4500$ is the same as in the step before. But we shall now be able to reach the maximum $f(10, 5) = 5500$ in the final step.

$$\mathbf{T}_3 = \begin{bmatrix} 1 & 0 & 0 & \frac{100}{3} & 0 & \frac{175}{3} & 5500 \\ 0 & 0 & 0 & 1 & 6 & -\frac{7}{2} & 30 \\ 0 & 2 & 0 & -\frac{1}{3} & 0 & \frac{2}{3} & 20 \\ 0 & 0 & 2 & \frac{1}{3} & 0 & -\frac{1}{6} & 10 \end{bmatrix} \quad \begin{matrix} \text{R1} + \frac{100}{3}\text{R2} \\ \text{R2} \\ \text{R3} - \frac{1}{3}\text{R2} \\ \text{R4} + \frac{1}{3}\text{R2} \end{matrix}$$

We see that $x_1 = 20/2 = 10$, $x_2 = 10/2 = 5$, $x_3 = 0$, $x_4 = 30/6 = 5$, $x_5 = 0$, $z = 5500$.

Problem 5 shows that the extra step (which gave no increase of $z = f(x)$) could have been avoided if we had chosen 4 (instead of 2) as the first pivot.

6. The maximum $f(0, 2.4, 0) = 2.4$ is obtained as follows.

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 4 & 5 & 8 & 1 & 0 & 12 \\ 0 & 8 & 5 & 4 & 0 & 1 & 12 \end{bmatrix}$$

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -\frac{3}{8} & -\frac{1}{2} & 0 & \frac{1}{8} & \frac{3}{2} \\ 0 & 0 & \frac{5}{2} & 6 & 1 & -\frac{1}{2} & 6 \\ 0 & 8 & 5 & 4 & 0 & 1 & 12 \end{bmatrix} \quad \begin{matrix} \text{R1} + \frac{1}{8}\text{R3} \\ \text{R2} - \frac{1}{2}\text{R3} \\ \text{R3} \end{matrix}$$

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{2}{5} & \frac{3}{20} & \frac{1}{20} & \frac{12}{5} \\ 0 & 0 & \frac{5}{2} & 6 & 1 & -\frac{1}{2} & 6 \\ 0 & 8 & 0 & -8 & -2 & 2 & 0 \end{bmatrix} \quad \begin{matrix} \text{R1} + \frac{3}{20}\text{R2} \\ \text{R2} \\ \text{R3 − 2 R2} \end{matrix}$$

From $\mathbf{T}_2$ we see that $x_1 = 0/8 = 0$, $x_2 = 6/\frac{5}{2} = 12/5$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, $z = 12/5$.

8. Maximize $\tilde{f} = -2x_1 + x_2$. The result is $-\tilde{f}_{\min} = \tilde{f}_{\max} = \tilde{f}(2, 3) = -1$, hence $f_{\min} = 1$. The calculation is as follows. An artificial variable x_6 is defined by

$$x_3 = -5 + x_1 + x_2 + x_6.$$

A corresponding objective function is

$$\tilde{\tilde{f}} = \tilde{f} - Mx_6 = (-2 + M)x_1 + (1 + M)x_2 - Mx_3 - 5M.$$

The corresponding matrix is

$$\mathbf{T}_0 = \begin{bmatrix} 1 & 2-M & -1-M & M & 0 & 0 & -5M \\ 0 & 1 & 1 & -1 & 0 & 0 & 5 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 5 & 4 & 0 & 0 & 1 & 40 \end{bmatrix}.$$

From this we obtain

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -3 & 2 & 0 & 0 & -10 \\ 0 & 1 & 1 & -1 & 0 & 0 & 5 \\ 0 & 0 & 2 & -1 & 1 & 0 & 6 \\ 0 & 0 & -1 & 5 & 0 & 1 & 15 \end{bmatrix} \quad \begin{matrix} \text{R1} + (M-2)\text{R2} \\ \text{R2} \\ \text{R3} + \text{R2} \\ \text{R4} - 5\,\text{R2} \end{matrix}$$

and

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 2 \\ 0 & 0 & 2 & -1 & 1 & 0 & 6 \\ 0 & 0 & 0 & \frac{9}{2} & \frac{1}{2} & 1 & 18 \end{bmatrix} \quad \begin{matrix} \text{R1} + \frac{3}{2}\,\text{R3} \\ \text{R2} - \frac{1}{2}\,\text{R3} \\ \text{R3} \\ \text{R4} + \frac{1}{2}\,\text{R3} \end{matrix}$$

We see that $x_1 = 2/1 = 2$, $x_2 = 6/2 = 3$, $x_3 = 0$, $x_4 = 0$, $x_5 = 18$, $\tilde{\tilde{f}} = -1$.

10. An artificial variable x_6 is defined by

$$x_4 = x_1 + 2x_2 - 6 + x_6$$

and a corresponding objective function by

$$\begin{aligned} \hat{f} &= 2x_1 + x_2 - Mx_6 = 2x_1 + x_2 - M(x_4 - x_1 - 2x_2 + 6) \\ &= (2 + M)x_1 + (1 + 2M)x_2 - Mx_4 - 6M. \end{aligned}$$

This gives the matrix

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -2-M & -1-2M & 0 & M & 0 & -6M \\ 0 & 2 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 & 0 & 6 \\ 0 & 1 & 1 & 0 & 0 & 1 & 4 \end{bmatrix}$$

and from it

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -\frac{3}{2}M & 1+\frac{1}{2}M & M & 0 & 2-5M \\ 0 & 2 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -1 & 0 & 5 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 & 3 \end{bmatrix} \quad \begin{matrix} \text{R1} + (1+\frac{1}{2}M)\,\text{R2} \\ \text{R2} \\ \text{R3} - \frac{1}{2}\,\text{R2} \\ \text{R4} - \frac{1}{2}\,\text{R2} \end{matrix}$$

and from this

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 3M & 0 & 1+2M & M & 0 & 2-2M \\ 0 & 2 & 1 & 1 & 0 & 0 & 2 \\ 0 & -3 & 0 & -2 & -1 & 0 & 2 \\ 0 & -1 & 0 & -1 & 0 & 1 & 2 \end{bmatrix} \quad \begin{matrix} \text{R1} + \frac{3}{2}M\,\text{R2} \\ \text{R2} \\ \text{R3} - \frac{3}{2}\,\text{R2} \\ \text{R4} - \frac{1}{2}\,\text{R2} \end{matrix}$$

which still contains M.

SOLUTIONS TO CHAPTER 20 REVIEW, page 1007

12. 9 steps give the solution $[-1, 2]$ to 6S. Steps 1–5 give

x_1	x_2
−1.01462	3.77669
−0.888521	2.07432
−1.00054	2.06602
−0.995857	2.00276
−1.00002	2.00245

14. The values obtained are

x_1	x_2
−1.04366	0.231924
−0.758212	1.51642
−1.01056	1.5725
−0.941538	1.88308
−1.00255	1.89664

Gradients (times a scalar) are obtained by calculating differences of subsequent values. Orthogonality follows from the fact that we change direction when we are tangent to a level curve and then proceed perpendicular to it.

16. Replace $-\nabla f$ by ∇f.

24. $f_{\max} = f(6, 3) = 180$

CHAPTER 21 Graphs and Combinatorial Optimization

SECTION 21.1. Graphs and Digraphs, page 1010

Purpose. To explain the concepts of a graph and a digraph (directed graph) and related concepts, as well as their computer representations.

Main Content, Important Concepts

Graph, vertices, edges

Incidence of a vertex v with an edge, degree of v

Digraph

Adjacency matrix

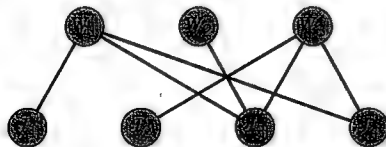
Incidence matrix

Vertex incidence list, edge incidence list

Comment on Content

Graphs and digraphs have become more and more important, due to an increase of supply and demand—a supply of more and more powerful methods of handling graphs and digraphs, and a demand for those methods in more and more problems and fields of application. Our chapter, devoted to the modern central area of combinatorial optimization, will give us a chance to get a feeling for the usefulness of graphs and digraphs in general.

SOLUTIONS TO PROBLEM SET 21.1, page 1014

4.

6. $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$ **8.** $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ **12.**

16. Join v_1 to $v_2, \cdots, v_n$, then v_2 to $v_3, \cdots, v_n$, then v_3 to $v_4, \cdots, v_n$, etc.; then take the sum $1 + 2 + \cdots + (n-1) = \frac{1}{2}n(n-1)$, the number of edges you have used in that process.

18. Vertex

$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

20.

Vertex	Incident Edges
1	$-e_1, -e_2, e_3, -e_4$
2	e_1
3	$e_2, -e_3$
4	e_4

SECTION 21.2. Shortest Path Problems. Complexity, page 1015

Purpose. To explain a method (by Moore) of determining a shortest path from a given vertex s to a given vertex t in a graph, all of whose edges have length 1.

Main Content, Important Concepts

Moore's algorithm (Table 21.1)

BFS (Breadth First Search), DFS (Depth First Search)

Complexity of an algorithm

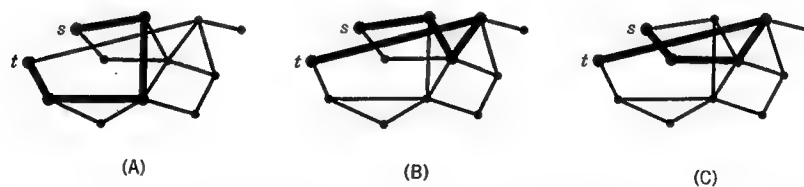
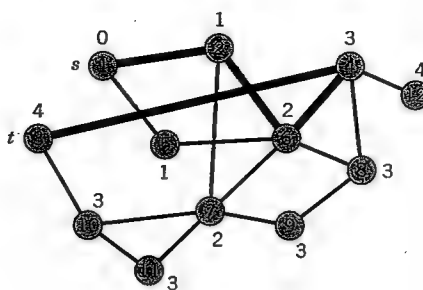
Efficient, polynomially bounded

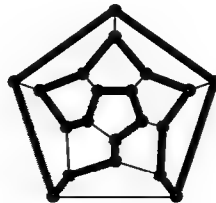
Comment on Content

The basic idea of Moore's algorithm is quite simple. A few related ideas and problems are illustrated in the problem set.

SOLUTIONS TO PROBLEM SET 21.2, page 1019

2. There are 3 shortest paths, of length 4 each:

(A) (B) (C)

Which one we obtain in backtracking depends on the numbering (not labeling!) of the vertices and on the backtracking rule. For the rule in Example 1 and the numbering shown in the following figure we get (B).

If we change the rule and let the computer look for largest (instead of smallest) numbers, we get (A).

4. $n - 1$. If it had more, a vertex would appear more than once and the corresponding cycle could be omitted. One edge.
6. This is true for $l = 0$ since then $v = s$. Let it be true for an $l - 1$. Then $\lambda(v_{l-1}) = l - 1$ for the predecessor v_{l-1} of v on a shortest path $s \to v$. We claim that when v_{l-1} gets labeled, v is still unlabeled (so that we shall have $\lambda(v) = l$ as wanted). Indeed, if v were labeled, it would have a label less than l, hence distance less than l by Prob. 5, contradicting that v has distance l.
8. No
10.

12. Delete the edge (2, 4).
14. $1 - 2 - 3 - 4 - 5 - 3 - 1, \quad 1 - 3 - 4 - 5 - 3 - 2 - 1$
16. Let T: $s \to s$ be a shortest postman trail and v any vertex. Since T includes each edge, T visits v. Let T_1: $s \to v$ be the portion of T from s to the *first* visit of v and T_2: $v \to s$ the other portion of T. Then the trail $v \to v$ consisting of T_2 followed by T_1 has the same length as T and solves the postman problem.

SECTION 21.3. Bellman's Optimality Principle. Dijkstra's Algorithm, page 1020

Purpose. This section extends the previous one to graphs whose edges have any (positive) length and explains a popular corresponding algorithm (by Dijkstra).

Main Content, Important Concepts

Bellman's optimality principle, Bellman's equations

Dijkstra's algorithm (Table 21.2)

Comment on Content

Throughout this chapter, one should emphasize that algorithms are needed because most practical problems are so large that solution by inspection would fail, even if one were satisfied with approximately optimal solutions.

SOLUTIONS TO PROBLEM SET 21.3, page 1023

2. Let j be the vertex that gave k its present label L_k, namely, $L_j + l_{jk}$. After this label was assigned, j did not change its label, since it was then removed from $\mathcal{TL}$. Next, find the vertex that gave j its permanent label, etc. This backward search traces a path from 1 to k, whose length is exactly L_k.
4. The algorithm gives

   1. $L_1 = 0, \tilde{L}_2 = 2, \tilde{L}_3 = 6, \tilde{L}_4 = 8, \tilde{L}_5 = \infty$
   2. $L_2 = 2, k = 2$

3. $\tilde{L}_3 = \min\{6, 2 + l_{23}\} = 5$
   $\tilde{L}_4 = \min\{8, 2 + l_{24}\} = 8$
   $\tilde{L}_5 = \min\{\infty, 2 + \infty\} = \infty$
2. $L_3 = 5, k = 3$
3. $\tilde{L}_4 = \min\{8, 5 + l_{34}\} = 8$
   $\tilde{L}_5 = \min\{\infty, 5 + \infty\} = \infty$
2. $L_4 = 8, k = 4$
3. $\tilde{L}_5 = \min\{\infty, 8 + l_{45}\} = 28$
2. $L_5 = 28, k = 5,$
   so that the answer is
   (1, 2), (1, 4), (2, 3), (4, 5); $L_2 = 2, L_3 = 5, L_4 = 8, L_5 = 28.$

6. Dijkstra's algorithm gives
1. $L_1 = 0, \tilde{L}_2 = 15, \tilde{L}_3 = 2, \tilde{L}_4 = 10, \tilde{L}_5 = 6$
2. $L_3 = 2$
3. $\tilde{L}_2 = \min\{15, 2 + l_{32}\} = 15$
   $\tilde{L}_4 = \min\{10, 2 + l_{34}\} = 10$
   $\tilde{L}_5 = \min\{6, 2 + l_{35}\} = 5$
2. $L_5 = 5$
3. $\tilde{L}_2 = \min\{15, 5 + l_{52}\} = 15$
   $\tilde{L}_4 = \min\{10, 5 + l_{54}\} = 9$
2. $L_4 = 9$
3. $\tilde{L}_2 = \min\{15, 9 + l_{42}\} = 14$
2. $L_2 = 14.$

The answer is (1, 3), (2, 4), (3, 5), (4, 5); $L_2 = 14, L_3 = 2, L_4 = 9, L_5 = 5.$

8. Dijkstra's algorithm gives
1. $L_1 = 0, \tilde{L}_2 = 8, \tilde{L}_3 = 10, \tilde{L}_4 = \infty, \tilde{L}_5 = 5, \tilde{L}_6 = \infty$
2. $L_5 = 5$
3. $\tilde{L}_2 = \min\{8, 5 + l_{52}\} = 7$
   $\tilde{L}_3 = \min\{10, 5 + l_{53}\} = 10$
   $\tilde{L}_4 = \min\{\infty, 5 + l_{54}\} = 10$
   $\tilde{L}_6 = \min\{\infty, 5 + l_{56}\} = 7$
2. $L_2 = 7$
3. $\tilde{L}_3 = \min\{10, 7 + l_{23}\} = 9$
   $\tilde{L}_4 = \min\{10, 7 + l_{24}\} = 10$
   $\tilde{L}_6 = \min\{7, 7 + l_{26}\} = 7$
2. $L_6 = 7$
3. $\tilde{L}_3 = \min\{9, 7 + l_{63}\} = 9$
   $\tilde{L}_4 = \min\{10, 7 + l_{64}\} = 8$
2. $L_4 = 8$
3. $\tilde{L}_3 = \min\{9, 8 + l_{43}\} = 9$
2. $L_3 = 9.$

The answer is (1, 5), (2, 3), (2, 5), (4, 6), (5, 6); $L_2 = 7, L_3 = 9, L_4 = 8, L_5 = 5,$
$L_6 = 7.$

SECTION 21.4. Shortest Spanning Trees. Kruskal's Greedy Algorithm, page 1024

Purpose. After the discussion of shortest paths between two given vertices, this section is devoted to the construction of a tree in a given graph that is spanning (contains all vertices of the graph) and is of minimum length.

Main Content, Important Concepts

Tree

Cycle

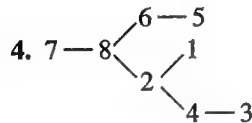
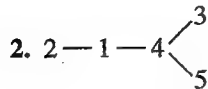
Kruskal's greedy algorithm (Table 21.3)

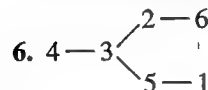
Comment on Content

Figure 459 illustrates that Kruskal's algorithm does not necessarily give a tree during each intermediate step, in contrast to another algorithm to be discussed in the next section.

SOLUTIONS TO PROBLEM SET 21.4, page 1027

```
             3
            /
2.  2—1—4
            \
             5
```

```
          6—5
         /      1
4.  7—8      /
         \  2
              \
               4—3
```

Note that trees, just as general graphs, can be sketched in different ways.

```
          2—6
         /
6.  4—3
         \
          5—1
```

8. Order the edges in descending order of length and delete them in this order, retaining an edge only if it would lead to the omission of a vertex or to a disconnected graph.

10. Order the edges in descending order of length and choose them in this order, rejecting an edge when a cycle would arise.

14. Let P_1: $u \to v$ and P_2: $u \to v$ be different. Let $e = (w, x)$ be in P_1 but not in P_2. Then P_1 without e together with P_2 is a connected graph. Hence it contains a path P_3: $w \to x$. Hence P_3 together with e is a cycle in T, a contradiction.

16. True for $n = 2$. Assume truth for all trees with less than n vertices. Let T be a tree with $n \geqq 2$ vertices, and (u, v) an edge of T. Then T without (u, v) contains no path $u \to v$, by Prob. 14. Hence this graph is disconnected. Let G_1, G_2 be its connected components, having n_1 and n_2 vertices, hence $n_1 - 1$ and $n_2 - 1$ edges, respectively, by the induction hypothesis, so that G has $n_1 - 1 + n_2 - 1 + 1 = n - 1$ edges.

18. Extend an edge e into a path by adding edges to its ends if such exist. A new edge attached at the end of the path introduces a new vertex, or closes a cycle, which is impossible. This extension terminates on both sides of e, yielding two vertices of degree 1.

20. If G is a tree, it has no cycles, and has $n - 1$ edges by Prob. 16. Conversely, let G have no cycles and $n - 1$ edges. Then G has 2 vertices of degree 1 by Prob. 18. Now prove connectedness by induction. True when $n = 2$. Assume true for $n = k - 1$. Let G with k vertices have no cycles and $k - 1$ edges. Omit a vertex v and its incident edge e, apply the induction hypothesis and add e and v back on.

SECTION 21.5. Prim's Algorithm for Shortest Spanning Trees, page 1028

Purpose. To explain another algorithm (by Prim) for constructing a shortest spanning tree in a given graph whose edges have arbitrary (positive) lengths.

Comments on Content

In contrast to Kruskal's greedy algorithm (Sec. 21.4), Prim's algorithm gives a tree at each intermediate step.

The problem set illustrates a few concepts that can be fit into the present cycle of ideas.

SOLUTIONS TO PROBLEM SET 21.5, page 1030

2. In Step 2 we first select a smallest l_{1j} for the $n - 1$ vertices outside U; these are $n - 2$ comparisons. Step 3 then requires $n - 2$ updatings (pairwise comparisons). In the next round we have $n - 3$ comparisons in Step 2 and $n - 3$ updatings in Step 3, and so on, until we finally end up with 1 comparison and 1 updating. The sum of all these numbers is $(n - 2)(n - 1) = O(n^2)$.

4. An algorithm for minimum spanning trees must examine each entry of the distance matrix at least once, because an entry not looked upon might have been one that should have been included in a shortest spanning tree. Hence, examining the relevant given information is already $O(n^2)$ work.

6. The algorithm gives

Vertex	Initial Label	Relabeling		
		(I)	(II)	(III)
2	$l_{12} = 6$	$l_{32} = 3$		
3	$l_{13} = 1$			
4	∞	$l_{34} = 10$	$l_{34} = 10$	$l_{54} = 2$
5	$l_{15} = 15$	$l_{15} = 15$	$l_{25} = 9$	

We see that we got

$$(1, 3), (3, 2), (2, 5), (5, 4); \quad L = 15.$$

The tree has the length $L = 15$.

To visualize the effect of the algorithm, use the graph (the figure) and for each step circle U and then go along the "circle" and look for the shortest edge that crosses it.

8. The algorithm gives

Vertex	Initial Label	Relabeling (I)	(II)	(III)	(IV)	(V)	(VI)
2	$l_{12} = 3$						
3	∞	$l_{23} = 4$					
4	∞	∞	$l_{34} = 3$	$l_{64} = 1$			
5	∞	∞	$l_{35} = 5$	$l_{35} = 5$	$l_{35} = 5$		
6	∞	$l_{26} = 10$	$l_{36} = 2$				
7	∞	$l_{27} = 7$	$l_{37} = 6$	$l_{37} = 6$	$l_{37} = 6$	$l_{37} = 6$	
8	$l_{18} = 8$	$l_{28} = 7$	$l_{28} = 7$	$l_{28} = 7$	$l_{28} = 7$	$l_{28} = 7$	$l_{28} = 7$

We see that we got

$$(1, 2), (2, 3), (3, 6), (6, 4), (3, 5), (3, 7)\ (2, 8).$$

The length is $L = 28$.

10. The algorithm proceeds as follows.

Vertex	Initial Label	Relabeling (I)	(II)	(III)	(IV)
2	$l_{12} = 20$	$l_{12} = 20$	$l_{32} = 4$	$l_{32} = 4$	
3	∞	$l_{53} = 6$			
4	∞	$l_{54} = 12$	$l_{34} = 2$		
5	$l_{15} = 8$				
6	$l_{16} = 30$	$l_{16} = 30$	$l_{16} = 30$	$l_{16} = 30$	$l_{26} = 10$

Hence we got successively

$$(1, 5), (5, 3), (3, 4), (3, 2), (2, 6); \quad L = 30.$$

In Prob. 6 of Sec. 21.4 we got the same edges, but in the order

$$(3, 4), (2, 3), (3, 5), (1, 5), (2, 6).$$

12. We obtain, in this order, the tree

$$(1, 2), (2, 8), (8, 7), (8, 6), (6, 5), (2, 4), (4, 3).$$

The length is 40.

14. TEAM PROJECT. (a) $\epsilon(1) = 16$, $\epsilon(2) = 22$, $\epsilon(3) = 12$.

(b) $d(G) = 24$, $r(G) = 12 = \epsilon(3)$, center $\{3\}$.

(c) 20, 14, center $\{3, 4\}$

(e) Let T^* be obtained from T by deleting all endpoints (= vertices of degree 1) together with the edges to which they belong. Since for fixed u, max $d(u, v)$ occurs only when v is an endpoint, $\epsilon(u)$ is one less in T^* than it is in T. Hence the vertices of minimum eccentricity in T are the same as those in T^*. Thus T has the same center as T^*. Delete the endpoints of T^* to get a tree T^{**} whose center is the same as that of T, etc. The process terminates when only one vertex or two adjacent vertices are left.

(f) Choose a vertex u and find a farthest v_1. From v_1 find a farthest v_2. Find w such that $d(w, v_1)$ is as close as possible to being equal to $\frac{1}{2}d(v_1, v_2)$.

SECTION 21.6. Networks. Flow Augmenting Paths, page 1031

Purpose. After shortest paths and spanning trees we discuss in this section a third class of practically important problems, the optimization of flows in networks.

Main Content, Important Concepts

Network, source, target (sink)

Edge condition, vertex condition

Path in a digraph, forward edge, backward edge

Flow augmenting path

Cut set, Theorems 1 and 2

Augmenting path theorem for flows

Max-flow min-cut theorem

Comment on Content

An algorithm for determining flow augmenting paths follows in the next section.

SOLUTIONS TO PROBLEM SET 21.6, page 1037

2. $T = \{2, 4, 6\}$, cap $(S, T) = 20 + 10 + 4 + 13 + 3 = 50$

4. $T = \{3, 4, 5, 6, 7\}$, cap $(S, T) = 7 + 8 = 15$

6. $T = \{4, 5, 6, 7\}$, cap $(S, T) = 7 + 10 = 17$

8. One is interested in flows *from s to t,* not in the opposite direction.

10. $\Delta_{14} = 6, \Delta_{45} = 3, \Delta_{52} = 1, \Delta_{23} = 3, \Delta_{36} = 7$

12. $\Delta_{12} = 5, \Delta_{24} = 8, \Delta_{45} = 2; \Delta_{12} = 5, \Delta_{25} = 3; \Delta_{13} = 4, \Delta_{35} = 9.$
From these numbers we see that flow augmenting paths are

$$P_1:\ 1 - 2 - 4 - 5,\ \Delta f = 2$$
$$P_2:\ 1 - 2 - 5,\ \Delta f = 3$$
$$P_3:\ 1 - 3 - 5,\ \Delta f = 4.$$

14. Flow augmenting paths are

$$P_1:\ 1 - 2 - 4 - 6,\ \Delta f = 1$$
$$P_2:\ 1 - 3 - 5 - 6,\ \Delta f = 1$$
$$P_3:\ 1 - 2 - 3 - 5 - 6,\ \Delta f = 1$$
$$P_4:\ 1 - 2 - 3 - 4 - 5 - 6,\ \Delta f = 1, \text{ etc.}$$

16. The maximum flow is $f = 14$. It can be realized by $f_{12} = 8, f_{13} = 6, f_{24} = 8, f_{43} = 4, f_{35} = 10, f_{45} = 4.$

18. The maximum flow is $f = 4$. It is realized by

$$f_{12} = 2, f_{13} = 2, f_{24} = 1, f_{23} = 1, f_{35} = 1, f_{34} = 2, f_{45} = 0,$$
$$f_{46} = 3, f_{56} = 1.$$

f is unique, but the way in which it is achieved is not, in general. In the present case we can change f_{45} from 0 to 1, f_{46} from 3 to 2, f_{56} from 1 to 2.

20. If $0 < f_{ij} < c$.

SECTION 21.7. Ford–Fulkerson Algorithm for Maximum Flow, page 1038

Purpose. To discuss an algorithm (by Ford and Fulkerson) for systematically increasing a flow in a network (e.g., the zero flow) by constructing flow augmenting paths until the maximum flow is reached.

Main Content, Important Concepts

Forward edge, backward edge

Ford–Fulkerson algorithm (Table 21.8)

Scanning of a labeled vertex

Comment on Content

Note that this is the first section in which we are dealing with digraphs.

SOLUTIONS TO PROBLEM SET 21.7, page 1040

2. Scanning the vertices in the order of their numbers, we get a flow augmenting path

$$P_1: \quad 1 - 2 - 4 - 6$$

with $\Delta_t = 1$ and then

$$P_2: \quad 1 - 3 - 4 - 6$$

with $\Delta_t = 1$, but no further flow augmenting path. Since the initial flow was 2, this gives the total flow $f = 4$.

4. The given flow equals 9. We first get the flow augmenting path

$$P_1: \quad 1 - 2 - 5 \quad \text{with} \quad \Delta_t = 2,$$

then the flow augmenting path

$$P_2: \quad 1 - 3 - 5 \quad \text{with} \quad \Delta_t = 5,$$

and finally the flow augmenting path

$$P_3: \quad 1 - 2 - 3 - 5 \quad \text{with} \quad \Delta_t = 1.$$

The maximum flow is $9 + 2 + 5 + 1 = 17$.

6. No. This follows from Theorem 4 in Sec. 21.6.

8. Not more work than in Example 1. Steps 1–7 are similar to those in the example and give the flow augmenting path

$$P_1: \quad 1 - 2 - 3 - 6,$$

which augments the flow from 0 to 11.

In determining a second flow augmenting path we scan 1, labeling 2 and 4 and getting $\Delta_2 = 9$, $\Delta_4 = 10$. In scanning 2, that is, trying to label 3 and 5, we cannot label 3 because $c_{ij} = c_{23} = f_{ij} = f_{23} = 11$, and we cannot label 5 because $f_{52} = 0$. In scanning 4 (i.e., labeling 5) we get $\Delta_5 = 7$. In scanning 5 we cannot label 3 because $f_{35} = 0$, and we further get $\Delta_6 = 3$. Hence a flow augmenting path is

$$P_2: \quad 1 - 4 - 5 - 6$$

and $\Delta_t = 3$. Together we get the maximum flow $11 + 3 = 14$ because no further flow augmenting paths can be found. The result agrees with that in Example 1.

10. The forward edges of the set are used to capacity; otherwise one would have been able to label their other ends. Similarly for the backward edges of the set, which carry no flow.

12. Let G have k edge-disjoint paths $s \to t$, and let $\tilde{f}$ be a maximum flow in G. Define on those paths a flow f by $f(e) = 1$ on each of their edges. Then $f = k \leqq \tilde{f}$ since $\tilde{f}$ is maximum. Now let G^* be obtained from G by deleting edges that carry no portion of $\tilde{f}$. Then, since each edge has capacity 1, there exist $\tilde{f}$ edge-disjoint paths in G^*, hence also in G, and $\tilde{f} \leqq k$. Together, $\tilde{f} = k$.

14. Since (S, T) is a cut set, there is no directed path $s \to t$ in G with the edges of (S, T) deleted. Since all edges have capacity 1, we thus obtain

$$\text{cap}\,(S, T) \geqq q.$$

Now let E_0 be a set of q edges whose deletion destroys all directed paths $s \to t$, and let G_0 denote G without these q edges. Let V_0 be the set of all those vertices v in G_0 for which there is a directed path $s \to v$. Let V_1 be the set of the other vertices in G. Then (V_0, V_1) is a cut set since $s \in V_0$ and $t \in V_1$. This cut set contains none of the edges of G_0, by the definition of V_0. Hence all the edges of (V_0, V_1) are in E_0, which has q edges. Now (S, T) is a *minimum* cut set, and all the edges have capacity 1. Thus,

$$\text{cap}\,(S, T) \leqq \text{cap}\,(V_0, V_1) \leqq q.$$

Together, $\text{cap}\,(S, T) = q$.

SECTION 21.8. Assignment Problems. Bipartite Matching, page 1041

Purpose. As the last class of problems, in this section we explain assignment problems (of workers to jobs, goods to storage spaces, etc.), so that the vertex set V of the graph consists of two subsets S and T and vertices in S are assigned (related by edges) to vertices in T.

Main Content, Important Concepts

Bipartite graph $G = (V, E) = (S, T; E)$

Matching, maximum cardinality matching

Exposed vertex

Alternating path, augmenting path

Matching algorithm (Table 21.9)

Comment on Content

A few additional problems on graphs, related to the present circle of ideas as well as of a more general nature, are contained in the problem set.

SOLUTIONS TO PROBLEM SET 21.8, page 1045

2. $S = \{1, 5\}$; $T = \{2, 3, 4\}$. Just move 2 down and you see it.

4. Yes, $S = \{1, 4, 5, 8\}$

6. No, as for a triangle, septangle, etc., whereas square, hexagon, octagon, · · · , are bipartite.

8. 1 – 2 – 3 – 7 – 5 – 4

10. (1, 4), (2, 3), (5, 7)

12. From the answer to Prob. 9 we see that as a matching of cardinality 3 we can take (1, 4), (3, 6), (7, 8). Addition of (2, 5) gives the desired matching of maximum cardinality 4.

14. 5. (Make a sketch.)

16.

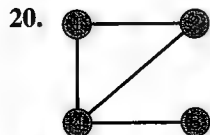
	Period			
	1	2	3	4
T_1	c_4	c_3	c_1	—
T_2	c_1	c_4	c_3	c_2
T_3	—	c_2	c_4	c_3

20. One might perhaps mention that the particular significance of K_5 and $K_{3,3}$ results from **Kuratowski's theorem,** stating that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (that is, it contains no subgraph obtained from K_5 or $K_{3,3}$ by subdividing the edges of these graphs by introducing new vertices on them).

SOLUTIONS TO CHAPTER 21 REVIEW, page 1046

16. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

20.

22.

Vertex	Incident Edges
1	e_1, e_3
2	e_1, e_2, e_4
3	e_2, e_5
4	e_3, e_4, e_5, e_6, e_7
5	e_6
6	e_7

24. (1, 2), (1, 4), (2, 3); $L_2 = 2, L_3 = 5, L_4 = 5$

28. The maximum flow is $f = 7$.

34. (1, 6), (4, 5), (2, 3), (7, 8)

PART G. PROBABILITY AND STATISTICS

CHAPTER 22 Data Analysis. Probability Theory

Change

The beginning is a new section on data analysis, explaining stem-and-leaf plots and boxplots and motivating probability by relative frequency.

SECTION 22.1. Data: Representation, Average, Spread, page 1050

Purpose. To discuss standard graphical representations of data in statistics. To introduce concepts that characterize the average size of the data values and their spread (their variability).

Main Content, Important Concepts

Stem-and-leaf plot

Histogram

Boxplot

Absolute frequency, relative frequency

Cumulative relative frequency

Outliers

Mean

Variance, standard deviation

Median, quartiles, interquartile range

Comment on Content

We explain the logic of the order of material. The graphical representations of data to be discussed in this section have become standard in connection with statistical methods. Average size and variability give the two most important general characterizations of data. Relative frequency will motivate probability as its theoretical counterpart. This is a main reason for presenting this material here before the beginning of our discussion of probability in this chapter. Randomness is not mentioned here because the introduction of samples (random samples) as a concept can wait until Chap. 23 when we shall need them in connection with statistical methods. The connection with this section will then be immediate and will provide no difficulty or duplication.

SOLUTIONS TO PROBLEM SET 22.1, page 1054

2. $q_L = 16$, $q_M = 17$, $q_U = 17.5$. Not symmetric with respect to q_M.

4. $q_L = -0.51$, $q_M = -0.18$, $q_U = 0.25$

6. $q_L = 11.0$, $q_M = 12.6$, $q_U = 13.4$

8. $q_L = 82$, $q_M = 84$, $q_U = 86$

10. $q_L = 199$, $q_M = 201$, $q_U = 201$

12. $\bar{x} = 16.9$, $s = 0.83$, $\text{IQR} = 1.5$

14. $\bar{x} = 12.6$ but $q_M = 7$. The data are not sufficiently symmetric. $s = 9.07$.

16. $x_{\min} \leqq x_j \leqq x_{\max}$. Now sum over j from 1 to n. Then divide by n to get $x_{\min} \leqq \bar{x} = x_{\max}$.

18. Points to consider are the amounts of calculation, the size of the data (in using quartiles we lose information—the larger the number of data points, the more information we lose), and the symmetry and asymmetry of the data. In the case of symmetry we have better agreement between quartiles on the one hand and mean and variance on the other, as in the case of data with considerable deviation from symmetry.

SECTION 22.2. Experiments, Outcomes, Events, page 1055

Purpose. To introduce basic concepts needed throughout Chaps. 22 and 23.

Main Content, Important Concepts

Experiment

Sample space S, outcomes, events

Union, intersection, complements of events

Mutually exclusive events

Representation of sets by Venn diagrams

Comment on Content

To make the chapter self-contained, we explain the modest amount of set-theoretical concepts needed in the next sections, although most students will be familiar with these matters.

SOLUTIONS TO PROBLEM SET 22.2, page 1057

2. $6^2 = 36$ outcomes, which are ordered pairs (1, 1), (1, 2), $\cdots$ (6, 6), where the first number refers to the first die and the second number to the second die.

4. Let A: *Six*, $N = A^C$: *No Six*. Then the infinitely many outcomes are A, NA, NNA, $NNNA$, etc.

6. No, $A \cap B = S \setminus (\{RRR\} \cup \{LLL\})$. Yes, we cannot obtain 2 right-handed and 2 left-handed screws in the same trial because we draw only 3 screws.

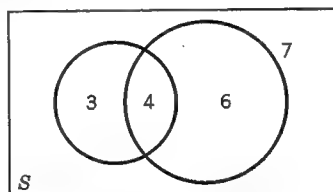
8. $A = \{(1, 1), \cdots, (6, 6)\}$, $B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$, $A \cap B = \{(1, 1), (2, 2)\}$, etc.

10. With the notation in Prob. 4 we have

$$E = \{A, NA, NNA, NNNA, NNNNA\}.$$

The complement is

E^C: *Rolling 6 or more times to get the first Six.*

12. T has 7 elements, L has 10, $T \cap L$ has 4.

Section 22.2. Problem 12

14. For instance, for the first formula we can proceed as follows (see the figure). On the right,

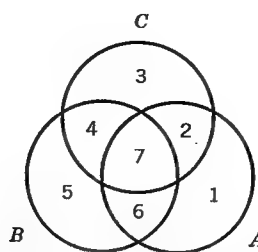
$$A \cup B\text{: All except 3}$$
$$A \cup C\text{: All except 5}$$

and the intersection of these two is

$$\text{Right side: All except 3 and 5.}$$

On the left,

$$A = 1 \cup 2 \cup 6 \cup 7$$
$$B \cap C = 4 \cup 7$$

and the union of these two gives the same as on the right.
Similarly for the other formula.

Section 22.2. Problem 14

SECTION 22.3. Probability, page 1058

Purpose. To introduce

1. Laplace's elementary probability concept based on *equally likely* outcomes,
2. The general probability concept defined axiomatically.

Main Content, Important Concepts

Definition 1 of probability

Definition 2 of probability

Motivation of the axioms of probability by relative frequency

Complementation rule, addition rules

Conditional probability

Multiplication rule, independent events

Sampling with and without replacement

Comments on Content

Whereas Laplace's definition of probability takes care of some applications and some statistical methods (for instance, nonparametric methods in Sec. 23.8), the major part of applications and theory will be based on the axiomatic definition of probability, which should thus receive the main emphasis in this section.

Sampling with and without replacement will be discussed in detail in Sec. 22.7.

SOLUTIONS TO PROBLEM SET 22.3, page 1063

2. $A^C = \{(5, 6), (6, 5), (6, 6)\}$, $P(A^C) = 3/36$. *Answer:* $1 - 3/36 = 11/12$

4. Increase. The probabilities are (20/30)(19/29) for *RR*, (20/30)(10/29) for *RL* and *LR*. For *RR* the probability decreases, whereas the other two probabilities increase. *Answer:* $780/870 = 0.89655 > 8/9 = 0.88889$

6. $P = 1/6 + 1/6 + 1/6 - 1/36 - 1/36 - 1/36 + 1/216 = 91/216$. Check by the complementation rule: $1 - 5^3/216 = 91/216$ because each of the dice can independently show one of the numbers $1, \cdots, 5$, whereas 6 is out.

8. $A^c = \{(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1)\}$. The sample space has $10 \cdot 10 = 100$ outcomes. *Answer:* $1 - 6/100 = 94\%$

10. By the multiplication rule (Theorem 4) we obtain

(a) $$\frac{100}{200} \cdot \frac{99}{199} = 24.874\%$$

(b) $$\frac{100}{200} \cdot \frac{100}{199} + \frac{100}{200} \cdot \frac{100}{199} = 50.25\%$$

(c) Same as (a).

Since (a)–(c) exhaust all possibilities, these probabilities must add up to 1, which provides a way of checking results in this and similar cases.

(d) $$\frac{50}{200} \cdot \frac{49}{199} = 6.16\%$$

12. $P^4 = 0.99$ gives $P = 0.99749$ as the probability that a single switch does not fail during a given time interval, and the *answer* is the complement of this, namely, 0.25%.

14. Drawing without replacement from the (hypothetically infinite) production that is going on. The probabilities are

(a) $0.98^2 = 96.04\%$
(b) $2 \cdot 0.98 \cdot 0.02 = 3.92\%$
(c) $0.02^2 = 0.04\%$

and the sum is 1.

16. We list the outcomes that favor the event whose probability we want to determine, and after each outcome the corresponding probability (F = female, M = male):

FF	1/4
MFF	1/8
FMF	1/8
MMFF	1/16
MFMF	1/16
FMMF	1/16

This gives the answer 11/16.

18. We have

$$A = B \cup (A \cap B^C)$$

where B and $A \cap B^C$ are disjoint because B and B^C are disjoint. Hence by Axiom 3,

$$P(A) = P(B) + P(A \cap B^C) \geqq P(B)$$

because $P(A \cap B^C)$ is a probability, hence nonnegative.

20. We have

$$P(A) = 2/4 = 1/2, \qquad P(B) = 1/2, \qquad P(C) = 1/2$$

and

$$P(A \cap B) = 1/4, \qquad P(B \cap C) = 1/4, \qquad P(C \cap A) = 1/4,$$

but $P(A \cap B \cap C) = 0$ because there is no chip numbered 111; hence

$$P(A \cap B \cap C) \neq P(A)P(B)P(C) = 1/8.$$

SECTION 22.4. Permutations and Combinations, page 1064

Purpose. To discuss permutations and combinations as tools necessary for systematic counting in experiments with a large number of outcomes.

Main Content

Theorems 1–3 contain the main properties of permutations and combinations we must know.

Formulas (5)–(14) contain the main properties of factorials and binomial coefficients we need in practice.

Comment on Content

The student should become aware of the surprisingly large size of the numbers involved in (1)–(4), even for relatively modest numbers n of given elements, a fact that would make attempts to list cases a very impractical matter.

SOLUTIONS TO PROBLEM SET 22.4, page 1068

2. The $5!/3! = 120/6 = 20$ permutations are

·	*ae*	*ai*	*ao*	*au*
ea	·	*ei*	*eo*	*eu*
ia	*ie*	·	*io*	*iu*
oa	*oe*	*oi*	·	*ou*
ua	*ue*	*ui*	*uo*	·

The $\binom{5}{3} = 10$ combination without repetition are obtained from the previous list by regarding the two pairs consisting of the same two letters (in opposite orders) as equal. The $\binom{5+2-1}{2} = \binom{6}{2} = 15$ combinations with repetitions consist of the 10 combinations just mentioned plus the 5 combinations

$$aa \quad ee \quad ii \quad oo \quad uu.$$

4. In $7! = 5040$ ways

6. In $\binom{8}{3}\binom{6}{2}\binom{5}{2} = 8400$ ways

8. There are $\binom{100}{10}$ samples of 10 from 100; hence the probability of picking a partic-

ular one is $1/\binom{100}{10}$. Now the number of samples containing the 3 male mice is $\binom{97}{7}$ because these are obtained by picking the 3 male mice and then 7 female mice from 97, which can be done in $\binom{97}{7}$ ways. Hence the answer is

$$\binom{97}{7}\Big/\binom{100}{10} = \frac{2}{2695} = 0.074\%.$$

10. In $6!/6 = 120$ ways

12. (a) 1/84, (b) 5/21

14. The complementary event (no two people have a common birthday) has probability

$$\frac{1}{365^{20}}\, 365 \cdot 364 \cdots 346 = 0.5886$$

(which can also be nicely computed by the Stirling formula). This gives the *answer* 41%, which is surprisingly large.

16. TEAM PROJECT. (a) There are n choices for the first thing and we terminate with the kth thing, for which we have $n - k + 1$ choices.

(b) The theorem holds when $k = 1$. Assuming that it holds for any fixed positive k, we show that the number of combinations of $(k + 1)$th order is $\binom{n+k}{k+1}$. From the assumption it follows that there are $\binom{n+k-1}{k}$ combinations of $(k + 1)$th order whose first element is 1 (this is the number of combinations of kth order). Then there are $\binom{n+k-2}{k}$ combinations of $(k + 1)$th order whose first element is 2 (this is the number of combinations of kth order of the $n - 1$ elements $2, 3, \cdots, n$). Then there are $\binom{n+k-3}{k}$ combinations of $(k + 1)$th order whose first element is 3, etc., and, by (13),

$$\binom{n+k-1}{k} + \binom{n+k-2}{k} + \cdots + \binom{k}{k} = \sum_{s=0}^{n-1}\binom{k+s}{k} = \binom{n+k}{k+1}.$$

(d) $a^k b^{n-k}$ is obtained by picking k of the n factors

$$(a + b)(a + b) \cdots (a + b) \qquad (n \text{ factors})$$

and choosing a from each of k factors (and b from the remaining $n - k$ factors); by Theorem 3, this can be done in $\binom{n}{k}$ ways.

(e) Apply the binomial theorem to

$$(1 + b)^p(1 + b)^q = (1 + b)^{p+q}.$$

b^r has the coefficient $\binom{p+q}{r}$ on the right and $\sum \binom{p}{k}\binom{q}{r-k}$ on the left.

SECTION 22.5. Random Variables, Probability Distributions, page 1069

Purpose. To introduce the concepts of discrete and continuous random variables and their distributions (to be followed up by the most important special distributions in Secs. 22.7 and 22.8).

Main Content, Important Concepts

Random variable X, distribution function $F(x)$

Discrete random variable, its probability function

Continuous random variable, its density

Comments on Content

The definitions in this section are general, but the student should not be scared because the number of distributions one needs in practice is small, as we shall see.

Discrete random variables occur in experiments in which we ***count,*** continuous random variables in experiments in which we ***measure.***

For both kinds of random variables X the definition of the distribution function $F(x)$ is the same, namely, $F(x) = P(X \leqq x)$, so that it permits a uniform treatment of all X. For discrete X the function $F(x)$ is piecewise constant; for continuous X it is continuous. For obtaining an impression of the distribution of X the probability function or density is more useful than $F(x)$.

SOLUTIONS TO PROBLEM SET 22.5, page 1074

2. $k = 1/8$ because of (6) and $1 + 3 + 3 + 1 = 8$

4. $k = 1/100$ because of (6) and $1 + 8 + 27 + 64 = 100$

6. $F = 0, -\frac{1}{2} + \frac{1}{4}x, 1$ if $x < 2, 2 < x < 6, x > 6$, respectively. The probabilities are $\frac{1}{2}$ and $\frac{1}{4}$.

This problem and Prob. 7 are important to the student in explaining the two basic tasks.

1. Find P for given x,
2. Find $x = c$ for given P

in the simplest possible situation.

8. $f = 0.1e^{-0.1x}$, $1 - e^{-0.1x} = 0.95$, $x = (\ln 20)/0.1 = 29.96$

10. 42/90, 42/90, 6/90, 0, 48/90

12. To have the area under the density curve equal to 1, we must have $k = 5$. If A denotes *"Defective"* and $B = A^C$, we have

$$P(B) = 5\int_{119.91}^{120.09} dx = 0.9, \qquad P(A) = 0.1,$$

so that about 50 of the 500 axles will be defective. This can also be seen without calculation.

14. The outcomes and their probabilities are (A: Six, $B = A^C$)

$$P(A) = \tfrac{1}{6}$$
$$P(BA) = \tfrac{5}{6} \cdot \tfrac{1}{6}$$
$$P(BBA) = \tfrac{5}{6} \cdot \tfrac{5}{6} \cdot \tfrac{1}{6}, \quad \text{etc.}$$

Hence the event $X = x$: *First Six in rolling x times* has the probability

$$f(x) = \frac{1}{6}\left(\frac{5}{6}\right)^{x-1}, \qquad x = 1, 2, \cdots.$$

We can now verify (6) by applying the sum formula of the geometric series:

$$\sum_{x=1}^{\infty} f(x) = \frac{1}{6}\sum_{x=1}^{\infty}\left(\frac{5}{6}\right)^{x-1} = \frac{1}{6}\sum_{y=0}^{\infty}\left(\frac{5}{6}\right)^{y} = \frac{\frac{1}{6}}{1-\frac{5}{6}} = 1.$$

Here $x - 1 = y$.

16. Integrating the density, we obtain the distribution function

$$F(x) = 0 \text{ if } x < -1,\ F(x) = \tfrac{1}{2}(x+1)^2 \text{ if } -1 \leqq x < 0,$$
$$F(x) = 1 - \tfrac{1}{2}(x-1)^2 \text{ if } 0 \leqq x < 1,\ F(x) = 1 \text{ if } x \geqq 1.$$

About 500 of the cans will contain 100 gallons or more because $Y = 100$ corresponds to $X = 0$ and $F(0) = 0.5$. Similarly, $Y = 99.5$ corresponds to $X = -0.5$ and $F(-0.5) = 0.125$; this is the probability that a can will contain less than 99.5 gallons. Finally, $F(-1) = 0$ is the answer to the last question.

18. By differentiation,

$$f(x) = 0.4x \quad \text{if } 2 < x < 3, \qquad f(x) = 0 \quad \text{otherwise.}$$

Furthermore,

$$P(2.5 < X \leqq 5) = F(5) - F(2.5) = 1 - 0.45 = 0.55,$$

that is, 55%.

20. $P(X \leqq c) = P(X \leqq b) + P(b < X \leqq c) \geqq P(X \leqq b)$ because all probabilities are nonnegative.

SECTION 22.6. Mean and Variance of a Distribution, page 1075

Purpose. To introduce the two most important parameters of a distribution, the *mean* μ of X (also called *expectation* of X), which measures the central location of the values of X, and the *variance* σ^2 of X, which measures the spread of those values.

Main Content, Important Concepts

Mean μ given by (1)

Variance σ^2 given by (2), standard deviation σ

Standardized random variable (6)

Short Courses. Mention definitions of mean and variance and go on to the special distributions in the next two sections.

Comments on Content

Important practical applications follow in Secs. 22.7, 22.8, and later.

The transformation theorem (Theorem 2) will be basic in Sec. 22.8 and will have various applications in Chap. 23.

Moments (8) and (9) will play no great role in our further work, but would be more important in a more theoretical approach on a higher level. We shall use them in Sec. 23.2.

SOLUTIONS TO PROBLEM SET 22.6, page 1078

2. $\mu = 3.5$, as follows by symmetry, without calculation. The variance is

$$\sigma^2 = \tfrac{1}{6}(2 \cdot 2.5^2 + 2 \cdot 1.5^2 + 2 \cdot 0.5^2) = 2.916667.$$

4. $\mu = 1,\ \sigma^2 = \int_0^\infty (x-1)^2 e^{-x}\, dx = 1$

6. $\mu = 2,\ \sigma^2 = 4^2 \cdot 1 = 16$

8. 20.8% because for a nondefective bolt we obtain

$$k\int_{1-0.06}^{1+0.06} f(x)\, dx = 750 \cdot 2\int_0^{0.06} (0.1 + z)(0.1 - z)\, dz = 0.792$$

where $x - 1 = z$.

10. About 70

12. We are asking for the sale x such that $F(x) = 0.95$. Integration of $f(x)$ gives

$$F(x) = 3x^2 - 2x^3 \qquad \text{if} \quad 0 \leqq x < 1.$$

From this we get the solution 0.8646, meaning that with a probability of 95% the sale will not exceed 8646 gallons (because here we measure in ten thousands of gallons). Thus

$$P(X \leqq 0.8646) = 0.95$$

and the complementary event that the sale will exceed 8646 gallons thus has a 5% chance,

$$P(X > 0.8646) = 0.05$$

and then the tank will be empty if it has a capacity of 8660 gallons.

14. He should pay the expected gain per game, which is the mean of $0.1\,X$, where X is the number that shows up; thus,

$$\frac{0.1}{6}\sum_{x=1}^{6} x = 35 \text{ cents.}$$

16. TEAM PROJECT. (a) $E(X - \mu) = E(X) - \mu E(1) = \mu - \mu = 0$. Furthermore,

$$\sigma^2 = E([X-\mu]^2) = E(X^2 - 2\mu X + \mu^2)$$
$$= E(X^2) - 2\mu E(X) + \mu^2 E(1),$$

where $E(X) = \mu$ and $E(1) = 1$, so that the result follows. The formula obtained has various practical and theoretical applications.

(b) $g(x) = X$ and the definition of expectation gives the defining formula for the mean. Similarly for (11). For $E(1)$ we get the sum of all possible values or the integral of the density taken over the x-axis, and in both cases the value is 1 because of (6) and (10) in Sec. 22.5.

(c) $E(X^k) = (b^{k+1} - a^{k+1})/[(b-a)(k+1)]$ by straightforward integration.

(d) Set $x - \mu = t$, write τ instead of t, set $\tau = -t$, and use $f(\mu - t) = f(\mu + t)$. Then

$$E([X-\mu]^3) = \int_{-\infty}^{\infty} t^3 f(\mu + t)\, dt = \int_{-\infty}^{0} \tau^3 f(\mu + \tau)\, d\tau + \int_0^\infty t^3 f(\mu + t)\, dt$$
$$= \int_\infty^0 (-t)^3 f(\mu - t)(-dt) + \int_0^\infty t^3 f(\mu + t)\, dt = 0.$$

(e) $\mu = 2$, $\sigma^2 = 2$, $\gamma = 4/2^{3/2} = \sqrt{2}$

(g) $f(1) = \frac{1}{2}$, $f(-4) = \frac{1}{3}$, $f(5) = \frac{1}{6}$. But for distributions of interest in applications, the skewness will serve its purpose.

SECTION 22.7. Binomial, Poisson, and Hypergeometric Distributions, page 1079

Purpose. To introduce the three most important discrete distributions and to illustrate them by typical applications.

Main Content, Important Concepts

Binomial distribution (2)–(4)

Poisson distribution (5), (6)

Hypergeometric distribution (8)–(10)

Short Courses. Discuss the binomial and hypergeometric distributions in terms of Examples 1 and 4.

Comments on Content

The "symmetric case" $p = q = 1/2$ of the binomial distribution with probability function (2*) is of particular practical interest. Formulas (3) and (4) will be needed from time to time. The approximation of the binomial distribution by the normal distribution follows in the next section.

SOLUTIONS TO PROBLEM SET 22.7, page 1083

2. $1 - 0.75^4 = 68.36\%$, where 0.75^4 is the probability of not hitting the target in the 4 trials.

4. $f(x) = \binom{100}{x} 0.04^x 0.96^{100-x} \approx 4^x e^{-4}/x!$. Values 0.018, 0.073, 0.147, 0.195, 0.195, 0.156. Sum 0.784. This leaves 21.6% for the remaining x-values.

6. $p = 0.02$, $q = 0.98$, hence $0.98^{15} = 74\%$

8. Let X be the number of customers per minute. The average number is $120/60 = 2$ per minute. Hence X has a Poisson distribution with mean 2. Waiting occurs if $X > 4$. The probability of the complement is $P(X \leqq 4) = 0.9473$ (see Table 6). Hence the answer is $1 - 0.9473 = 5\frac{1}{4}\%$.

10. For this problem, the hypergeometric distribution has the probability function

$$f(x) = \binom{5}{x}\binom{15}{3-x}\bigg/\binom{20}{3}.$$

The numerical values are

x	0	1	2	3
$P(X = x)$	$\frac{455}{1140}$	$\frac{525}{1140}$	$\frac{150}{1140}$	$\frac{10}{1140}$

These values sum up to 1, as they should.

12. If a package of $N = 100$ items contains precisely $M = 10$ defectives, then the probability that 10 items drawn without replacement contain no defectives is

$$f(0) = \binom{10}{0}\binom{90}{10}\Big/\binom{100}{10} = \frac{90 \cdot 89 \cdots 81}{100 \cdot 99 \cdots 91} = 33\%.$$

Answer: 67%, so the method is very poor.

14. TEAM PROJECT. (a) In each differentiation we get a factor x_j by the chain rule, so that

$$G^{(k)}(t) = \sum_j x_j^k e^{tx_j} f(x_j).$$

If we now set $t = 0$, the exponential function becomes 1 and we are left with the definition of $E(X^k)$. Similarly for a continuous random variable.

(d) By differentiation,

$$G'(t) = n(pe^t + q)^{n-1}pe^t,$$
$$G''(t) = n(n-1)(pe^t + q)^{n-2}(pe^t)^2 + n(pe^t + q)^{n-1}pe^t.$$

This gives, since $p + q = 1$,

$$E(X^2) = G''(0) = n(n-1)p^2 + np.$$

From this we finally obtain the desired result,

$$\sigma^2 = E(X^2) - \mu^2 = n(n-1)p^2 + np - n^2p^2 = npq.$$

(e) $G(t)$ gives $G(0) = 1$ and furthermore

$$G'(t) = e^{-\mu} \exp[\mu e^t]\, \mu e^t = \mu e^t G(t),$$
$$G''(t) = \mu e^t[G(t) + G'(t)],$$
$$E(X^2) = G''(0) = \mu + \mu^2,$$
$$\sigma^2 = E(X^2) - \mu^2 = \mu.$$

(f) By definition,

$$\mu = \sum x f(x) = \frac{1}{\binom{N}{n}} \sum x \binom{M}{x}\binom{N-M}{n-x}$$

(summation over x from 0 to n). Now

$$x\binom{M}{x} = \frac{xM(M-1)\cdots(M-x+1)}{x!}$$
$$= \frac{M(M-1)\cdots(M-x+1)}{(x-1)!} = M\binom{M-1}{x-1}.$$

Thus

$$\mu = \frac{M}{\binom{N}{n}} \sum \binom{M-1}{x-1}\binom{N-M}{n-x}.$$

Now (14), Sec. 22.4, is

$$\sum \binom{p}{k}\binom{q}{r-k} = \binom{p+q}{r}$$

(summation over k from 0 to r). With $p = M - 1$, $k = x - 1$, $q = N - M$,

$r - k = n - x$ we have $p + q = N - 1$, $r = k + n - x = n - 1$ and the formula gives

$$\mu = \frac{M}{\binom{N}{n}} \binom{N-1}{n-1} = n \frac{M}{N}.$$

SECTION 22.8. Normal Distribution, page 1085

Purpose. To discuss Gauss's normal distribution, the practically and theoretically most important distribution, and the practical use of the normal tables.

Main Content, Important Concepts

Normal distribution, its density (1) and distribution function (2)

Distribution function $\Phi(z)$, Tables A7, A8 in Appendix 5

De Moivre–Laplace limit theorem

Short Course. Emphasis on the use of Tables A7 and A8 in terms of some of the given examples and problems.

Comments on Content

Most important is that the student learn how to use Tables A7 and A8. Second, the student should get a feeling for the distribution of values as expressed in (6) or (7).

Applications of the De Moivre–Laplace theorem follow in Chap. 23.

Bernoulli's law of large numbers is included in the problem set.

SOLUTIONS TO PROBLEM SET 22.8, page 1090

2. From Table A7 in Appendix 5 we get

$$P(X \leqq 112.5) = F(112.5) = \Phi\left(\frac{112.5 - 105}{5}\right) = \Phi(1.5) = 0.9332,$$

$$P(X > 100) = 1 - F(100) = 1 - \Phi\left(\frac{100 - 105}{5}\right) = 1 - \Phi(-1)$$

$$= 1 - (1 - \Phi(1)) = 0.8413,$$

$$P(110.5 < X < 111.25) = \Phi\left(\frac{111.25 - 105}{5}\right) - \Phi\left(\frac{110.5 - 105}{5}\right)$$

$$= \Phi(1.25) - \Phi(1.1) = 0.8944 - 0.8643$$

$$= 0.0301.$$

4. We have

$$P(X \leqq c) = \Phi\left(\frac{c - 3.6}{0.1}\right) = 0.5.$$

Thus, from Table A8 in Appendix 5,

$$\frac{c - 3.6}{0.1} = 0, \qquad c = 3.6.$$

This could be seen without calculation. Next,

$$P(X > c) = 1 - \Phi\left(\frac{c - 3.6}{0.1}\right) = 0.1, \qquad \Phi\left(\frac{c - 3.6}{0.1}\right) = 0.9.$$

From the table,

$$\frac{c - 3.6}{0.1} = 1.282, \qquad c = 3.7282.$$

Finally,

$$P(-c < X - 3.6 \leqq c) = \Phi\left(\frac{c}{0.1}\right) - \Phi\left(\frac{-c}{0.1}\right) = 99.9\%,$$

$$\frac{c}{0.1} = 3.291, \qquad c = 0.3291.$$

6. Smaller. This should help the student in qualitative thinking and an understanding of standard deviation and variance.

8. We have $np = 4040 \cdot \frac{1}{2} = 2020$ and $\sigma = \sqrt{npq} = \sqrt{1010}$. By the De Moivre–Laplace theorem we thus obtain

$$\begin{aligned} P(2048 \leqq X \leqq 4040) &= \Phi\left(\frac{4040 - 2020 + 0.5}{\sqrt{1010}}\right) - \Phi\left(\frac{2048 - 2020 - 0.5}{\sqrt{1010}}\right) \\ &= 19.3\%. \end{aligned}$$

Hence the event actually observed has a not too small probability of occurring under the assumption that the coin was fair ($p = \frac{1}{2}$).

10. Applying the De Moivre–Laplace theorem, we get

$$\begin{aligned} P &= \sum_{x=0}^{10} \binom{1000}{x} 0.01^x 0.99^{1000-x} \\ &\approx \Phi\left(\frac{10 - 10 + 0.5}{\sqrt{9.9}}\right) - \Phi\left(\frac{0 - 10 - 0.5}{\sqrt{9.9}}\right) = 0.564. \end{aligned}$$

(The exact value is 0.583.)

12. We get the maximum load c from the condition

$$P(X \leqq c) = \Phi\left(\frac{c - 1500}{50}\right) = 5\%.$$

By Table A8 in Appendix 5,

$$\frac{c - 1500}{50} = -1.645, \qquad c = 1418 \text{ kg}.$$

14. TEAM PROJECT. (c) Let e denote the exponential function in (1). Then

$$(\sigma\sqrt{2\pi}f)'' = \left(-\frac{x-\mu}{\sigma^2}e\right)' = \left(-\frac{1}{\sigma^2} + \left(\frac{x-\mu}{\sigma^2}\right)^2\right)e = 0, \; (x-\mu)^2 = \sigma^2,$$

hence $x = \mu + \sigma$.

(d) Proceeding as suggested, we obtain

$$\begin{aligned} \Phi^2(\infty) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\,du \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-v^2/2}\,dv \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2/2} e^{-v^2/2}\,du\,dv = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2}\, r\,dr\,d\theta. \end{aligned}$$

The integral over θ equals 2π, which cancels the factor in front, and the integral over r equals 1, which proves the desired result.

(e) Writing β instead of σ in (1) and using $(x - \mu)/\beta = u$ and $dx = \beta\, du$, we obtain

$$\sigma^2 = \frac{1}{\sqrt{2\pi}\,\beta}\int_{-\infty}^{\infty}(x-\mu)^2 \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\beta}\right)^2\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\,\beta}\int_{-\infty}^{\infty}\beta^2 u^2 e^{-u^2/2}\beta\, du = \frac{\beta^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u^2 e^{-u^2/2}\, du$$

$$= \frac{\beta^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-u)(-ue^{-u^2/2})\, du \qquad \left(u = \frac{x-\mu}{\beta}\right)$$

$$= \beta^2\left[\frac{1}{\sqrt{2\pi}}(-u)e^{-u^2/2}\Big|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-u^2/2}\, du\right] = \beta^2.$$

(f) We have

$$P\left(\left|\frac{X}{n} - p\right| < \epsilon\right) = P([p - \epsilon]n < X < [p + \epsilon]n)$$

and apply (11) with $a = (p - \epsilon)n$, $b = (p + \epsilon)n$. Then, since np cancels,

$$\beta = (\epsilon n + 0.5)/\sqrt{npq}, \qquad \alpha = -\beta,$$

and $\alpha \to -\infty$, $\beta \to \infty$ as $n \to \infty$. Hence the above probability approaches $\Phi(\infty) - \Phi(-\infty) = 1 - 0 = 1$.

(g) Set $x^* = c_1 x + c_2$. Then $(x - \mu)/\sigma = (x^* - \mu^*)/\sigma^*$ and

$$\begin{aligned} F(x^*) &= P(X^* \leqq x^*) \\ &= P(X \leqq x) \\ &= \Phi((x - \mu)/\sigma) \\ &= \Phi((x^* - \mu^*)/\sigma^*). \end{aligned}$$

SECTION 22.9. Distributions of Several Random Variables, page 1091

Purpose. To discuss distributions of two-dimensional random variables, with an extension to n-dimensional random variables near the end of the section.

Main Content, Important Concepts

Discrete two-dimensional random variables and distributions

Continuous two-dimensional random variables and distributions

Marginal distributions

Independent random variables

Addition of means and variances

Short Courses. Omit this section. (Use the addition theorems for means and variances in Chap. 23 without proof.)

Comments on Content

The addition theorems (Theorems 1 and 3) resulting from the present material will be needed in Chap. 23; this is the main reason for the inclusion of this section.

Note well that the addition theorem for variances holds for *independent* random variables only. In contrast, the addition of means is true without that condition.

SOLUTIONS TO PROBLEM SET 22.9, page 1099

2. The answers are 0 and 1/32. Since the density is constant in that triangle, these results can be seen from a sketch of the triangle and the regions determined by the inequalities $x > 4$, $y > 4$ and $x \leqq 1$, $y \leqq 1$, respectively, without any integrations.

4. We have to integrate $f(x, y) = 1/32$ over y from 0 to $8 - x$, where this upper integration limit follows from $x + y = 8$. This gives the density of the desired marginal distribution in the form

$$f_1(x) = \int_0^{8-x} \frac{1}{32}\, dy = \frac{1}{4} - \frac{1}{32}x \quad \text{if } 0 \leqq x \leqq 8 \quad \text{and} \quad 0 \text{ otherwise.}$$

6. By Theorem 1 the mean is $10\,000 \cdot 2 = 20$ kg. By Theorem 3, assuming independence (which is reasonable), we find the variance $10\,000 \cdot 0.03^2 = 9$, hence the standard deviation 3 grams. Note that the mean is multiplied by $n = 10\,000$, whereas the standard deviation is multiplied only by $\sqrt{n} = 100$.

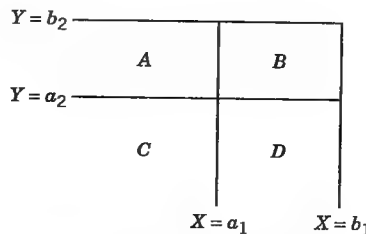
8. From the given distributions we obtain

$$f_1(x) = 25 \quad \text{if } 0.98 < x < 1.02 \quad \text{and} \quad 0 \text{ otherwise,}$$
$$f_2(y) = 25 \quad \text{if } 1.00 < y < 1.04 \quad \text{and} \quad 0 \text{ otherwise.}$$

A pin fits the hole if $X < 1$ and $P(X < 1) = 50\%$.

10. By Theorem 1 the mean is 105 lb. By Theorem 3, assuming independence, we get the variance $0.04 + 0.25$, hence the standard deviation $\sqrt{0.29} = 0.539$ lb.

12. No. Whereas for the mean it is *not* essential that the trials are not independent and one still obtains from $\mu = M/N$ (single trial) the result $\mu = nM/N$ (n trials) via Theorem 1, one cannot use Theorem 3 here; indeed, the variance $\sigma^2 = M(N - M)/N^2$ (single trial) does not lead to (10), Sec. 22.7.

14. (X, Y) takes a value in A, B, C, or D (see the figure) with probability $F(b_1, b_2)$, a value in A or C with probability $F(a_1, b_2)$, a value in C or D with probability $F(b_1, a_2)$, a value in C with probability $F(a_1, a_2)$, hence a value in B with probability given by the right side of (2).

Section 22.9. Problem 14

16. In the continuous case, (18) is obtained from (17) by differentiation, and (17) is obtained from (18) by integration. In the discrete case the proof results from the following theorem. Two random variables X and Y are independent if and only if the events of the form $a_1 < X \leqq b_1$ and $a_2 < Y \leqq b_2$ are independent. This theorem can be proved as follows. From (2), Sec. 22.5, we have

$$P(a_1 < X \leqq b_1)P(a_2 < Y \leqq b_2) = [F_1(b_1) - F_1(a_1)][F_2(b_2) - F_2(a_2)].$$

In the case of independence of the variables X and Y we conclude from (17) that the expression on the right equals

$$F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2).$$

Hence, by (2),

$$P(a_1 < X \leqq b_1)P(a_2 < Y \leqq b_2) = P(a_1 < X \leqq b_1, a_2 < Y \leqq b_2).$$

This means independence of $a_1 < X \leqq b_1$ and $a_2 < Y \leqq b_2$; see (14), Sec. 22.3. Conversely, suppose that the events are independent for any a_1, b_1, a_2, b_2. Then

$$P(a_1 < X \leqq b_1)P(a_2 < Y \leqq b_2) = P(a_1 < X \leqq b_1, a_2 < Y \leqq b_2).$$

Let $a_1 \to -\infty$, $a_2 \to -\infty$ and set $b_1 = x$, $b_2 = y$. This yields (17), that is, X and Y are independent.

SOLUTIONS TO CHAPTER 22 REVIEW, page 1100

26. $Q_L = 110$, $Q_M = 112$, $Q_U = 115$

28. $\bar{x} = 111.9$, $s = 4.0125$, $s^2 = 16.1$

30. $x_{\min} \leqq x_j \leqq x_{\max}$. Sum over j from 1 to n to get

$$n\,x_{\min} \leqq \sum_{j=1}^{n} x_j \leqq n\,x_{\max}.$$

Divide by n.

32. *HHH, HHT, HTH, THH, HTT, THT, TTH, TTT*

34. Obviously, $A \subseteq B$ implies $A \cap B = A$. Conversely, if $A \cap B = A$, then every element of A must also be in B, by the definition of intersection; hence $A \subseteq B$.

36. (a) $\binom{12}{3} = 220$, (b) $\binom{9}{3} = 84$, (c) $\binom{3}{1}\binom{9}{2} = 108$,

(d) $\binom{3}{2}\binom{9}{1} = 27$, (e) $\binom{3}{3} = 1$. Note that the sum of (b) through (e) is 220.

38. $f(x) = 2^{-x}$, $x = 1, 2, \cdots$. From this and the definition of mean we first have

$$\mu = \sum_{x=1}^{\infty} x \cdot 2^{-x} = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n.$$

This can be summed by the derivative of the geometric series with $q = \frac{1}{2}$, as follows.

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \qquad \sum_{n=1}^{\infty} nq^{n-1} = \frac{1}{(1-q)^2}.$$

Now multiply the last series by q to get $q/(1-q)^2$ on the right, and take $q = \frac{1}{2}$; then the right side equals 2 and the left side equals our series for the mean. Hence the *answer* is $\mu = 2$.

40. 6^6 outcomes. The $6!$ permutations of 1, 2, 3, 4, 5, 6 are of the desired type. *Answer:* $6!/6^6 = 5/324 = 1.5\%$

44. $1, \frac{1}{2}$

46. We first need

$$\mu = 2\int_0^1 x(1-x)\,dx = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

In the further integrations we can use the defining integrals of $E[X - \mu)^2]$ and $E[(X - \mu)^3]$ or, more simply,

$$\sigma^2 = E(X^2) - \mu^2 = 2\int_0^1 x^2(1-x)\,dx - \frac{1}{9} = 2\left[\frac{1}{3} - \frac{1}{4}\right] - \frac{1}{9} = \frac{1}{18}$$

and similarly,

$$\begin{aligned} E[(X-\mu)^3] &= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3 \\ &= E(X^3) - 3\mu E(X^2) + 2\mu^3 \\ &= 2\int_0^1 x^3(1-x)\,dx - 3\cdot\frac{1}{3}\cdot\frac{1}{6} + \frac{2}{27} \\ &= 2\left[\frac{1}{4} - \frac{1}{5}\right] - \frac{1}{6} + \frac{2}{27} = \frac{1}{135}. \end{aligned}$$

This gives

$$\gamma = \frac{18\sqrt{18}}{5\cdot 27} = \frac{2\sqrt{2}}{5}.$$

48. 0.1587, 0.6306, 0.5, 0.4950

50. 25.71 cm, 0.0205 cm

CHAPTER 23 Mathematical Statistics

Changes

The first section on random sampling is new. At the end of the chapter an introduction to correlation analysis has been added.

SECTION 23.1. Introduction. Random Sampling, page 1104

Purpose. To explain the role of (random) samples from populations.

Main Content, Important Concepts

Population

Sample

Random numbers, random number generator

Sample mean: $\bar{x}$; see (1)

Sample variance s^2; see (2)

Comments on Content

Sample mean and sample variance are the two most important parameters of a sample. $\bar{x}$ measures the central location of the sample values and s^2 their spread (their variability). Small s^2 may indicate high quality of production, high accuracy of measurement, etc.

Note well that $\bar{x}$ and s^2 will generally vary from sample to sample taken from the same population, whose mean μ and variance σ^2 are unique, of course. This is an important conceptual distinction that should be mentioned explicitly to the students.

SECTION 23.2. Estimation of Parameters, page 1106

Purpose. As a first statistical task we discuss methods for obtaining approximate values of unknown population parameters from samples; this is called *estimation of parameters.*

Main Content, Important Concepts

Point estimate, interval estimate

Method of moments

Maximum likelihood method

SOLUTIONS TO PROBLEM SET 23.2, page 1108

4. $l = 1/(b - a)^n$ is maximum if $b - a$ is as small as possible, that is, a equal to the smallest sample value and b equal to the largest.

6. $\mu = 1/\theta$, $\hat{\mu} = \bar{x}$

8. $\hat{\theta} = 1/\bar{x} = 2$, $F(x) = 1 - e^{-2x}$ if $x \geqq 0$ and 0 otherwise. A graph shows that the step function $\widetilde{F}(x)$ (the sample distribution function) approximates $F(x)$ reasonably well. (For goodness of fit, see Sec. 23.7.)

10. The likelihood function is (we can drop the binomial factors)

$$l = p^{k_1}(1-p)^{n-k_1}\cdots p^{k_m}(1-p)^{n-k_m}$$
$$= p^{k_1+\cdots+k_m}(1-p)^{nm-(k_1+\cdots+k_m)}.$$

The logarithm is

$$\ln l = (k_1 + \cdots + k_m)\ln p + [nm - (k_1 + \cdots + k_m)]\ln(1-p).$$

Equating the derivative with respect to p to zero, we get

$$(k_1 + \cdots + k_m)\frac{1}{p} = [nm - (k_1 + \cdots + k_m)]\frac{1}{1-p}.$$

Multiplication by $p(1-p)$ gives

$$(k_1 + \cdots + k_m)(1-p) = [nm - (k_1 + \cdots + k_m)]p.$$

By simplification,

$$k_1 + \cdots + k_m = nmp.$$

The result is

$$\hat{p} = \frac{1}{nm}\sum_{i=1}^{m} k_i.$$

12. The likelihood function is

$$l = f = p(1-p)^{x-1}.$$

The logarithm is

$$\ln l = \ln p + (x-1)\ln(1-p).$$

Differentiating and equating the derivative to zero, we get

$$\frac{1}{p} - \frac{x-1}{1-p} = 0.$$

Hence the *answer* is

$$\hat{p} = \frac{1}{x}.$$

14. $\hat{p} = 2/(7+6) = 2/13$, by Prob. 13.

SECTION 23.3. Confidence Intervals, page 1109

Purpose. To obtain interval estimates ("confidence intervals") for unknown population parameters for the normal distribution and other distributions.

Main Content, Important Concepts

Confidence interval for μ if σ^2 is known

Confidence interval for μ if σ^2 is unknown

t-distribution, its occurrence (Theorem 2)

Confidence interval for σ^2

Chi-square distribution, its occurrence (Theorem 3)

Distribution of a sum of independent normal random variables

Central limit theorem

Comments on Content

The present methods are designed for the normal distribution, but the central limit theorem permits their extension to other distributions, provided we have available sufficiently large samples.

The theorems giving the theory underlying the present methods also serve as the theoretical basis of tests in the next section. Hence these theorems are of basic importance.

We see that, although our task is the development of methods for the normal distribution, other distributions (t and chi-square) appear in the mathematical foundation of those methods.

SOLUTIONS TO PROBLEM SET 23.3, page 1117

2. $2.576 \cdot 3/\sqrt{100} = 0.773$. Length increase by 30%. The shift of $\bar{x}$ causes a corresponding shift (3 units) of the interval.

4. $n = 8$, $c = 1.960$, $\bar{x} = 10.25$, $k = 0.832$, so that we obtain the confidence interval

$$\text{CONF}_{0.95}\{9.41 \leqq \mu \leqq 11.09\}.$$

6. Reduction of the sample size by a factor 4 corresponds to an increase of the length by a factor 2.

8. $n = 290$ gives $L/\sigma \approx 0.3$, hence $L \approx 0.18$, $L/2 \approx 0.09$, so that the confidence interval is

$$\text{CONF}_{0.99}\{16.21 \leqq \mu \leqq 16.39\}.$$

10. $n - 1 = 4$; $F(c) = 0.995$ gives $c = 4.60$. From the sample we compute

$$\bar{x} = 659.2, \qquad s^2 = 22.70.$$

Hence $k = 9.8$ in Table 23.2, 4th step. This gives the confidence interval

$$\text{CONF}_{0.99}\{649.4 \leqq \mu \leqq 669.0\}.$$

12. $n = 24\,000$, $\bar{x} = 12\,012$, $\hat{p} = \bar{x}/n = 0.5005$. Now the random variable

$$X = \textit{Number of heads in } 24\,000 \textit{ trials}$$

is approximately normal with mean $24\,000p$ and variance $24\,000p(1 - p)$. Estimators are

$$24\,000\hat{p} = 12\,012 \qquad \text{and} \qquad 24\,000\hat{p}(1 - \hat{p}) = 5999.99.$$

For the standardized normal random variable we get from Table A8 in Appendix 5 and $\Phi(c) = 0.995$ the value

$$c = 2.576 = \frac{c^* - 12\,012}{\sqrt{6000}}$$

and

$$c^* - 12\,012 = 2.576\sqrt{6000} = 199.5$$

so that

$$\text{CONF}_{0.99}\{11\,812 \leqq \mu \leqq 12\,212\}$$

and by division by n,

$$\text{CONF}_{0.99}\{0.492 \leqq p \leqq 0.509\}.$$

14. $n - 1 = 9$ degrees of freedom, $F(c_1) = 0.025$, $c_1 = 2.70$, $F(c_2) = 0.975$, $c_2 = 19.02$ from Table A10. From the sample,

$$\bar{x} = 253.5, \qquad 9s^2 = 54.5.$$

Hence $k_1 = 54.5/2.70 = 20.19$, $k_2 = 54.5/19.02 = 2.86$. From Table 23.3 we thus obtain the confidence interval

$$\text{CONF}_{0.95}\{2.8 \leqq \sigma^2 \leqq 20.2\}.$$

16. $n - 1 = 7$ degrees of freedom, $F(c_1) = 0.025$, $c_1 = 1.69$, $F(c_2) = 0.975$, $c_2 = 16.01$ from Table A10. From the sample,

$$\bar{x} = 17.7625, \qquad (n-1)s^2 = 7s^2 = 0.73875.$$

Hence $k_1 = 0.437$, $k_2 = 0.046$. The answer is

$$\text{CONF}_{0.95}\{0.046 \leqq \sigma^2 \leqq 0.437\}.$$

18. By Theorem 1 in this section and by Team Project 14(g) in Sec. 22.8, the distribution of $4X_1 - X_2$ is normal with mean $4 \cdot 16 - 12 = 52$ and variance $16 \cdot 8 + 2 = 130$.

20. By Theorem 1, the load Z is normal with mean $40N$ and variance $4N$, where N is the number of bags. Now

$$P(Z \leqq 2000) = \Phi\left(\frac{2000 - 40N}{2\sqrt{N}}\right) = 0.95$$

gives the condition

$$\frac{2000 - 40N}{2\sqrt{N}} \geqq 1.645$$

by Table A8. The *answer* is $N = 49$ (since N must be an integer).

SECTION 23.4. Testing of Hypotheses, Decisions, page 1118

Purpose. Our third big task is testing of hypotheses. This section contains the basic ideas and the corresponding mathematical formalism. Applications to further tasks of testing follow in Secs. 23.5–23.8.

Main Content, Important Concepts

Hypothesis (null hypothesis)

Alternative (alternative hypothesis), one- and two-sided

Type I error (probability α = significance level)

Type II error (probability β; $1 - \beta$ = power of a test)

Test for μ with known σ^2 (Example 2)

Test for μ with unknown σ^2 (Example 3)

Test for σ^2 (Example 4)

Comparison of means (Example 5)

Comparison of variances (Example 6)

Comment on Content

Special testing procedures based on the present ideas have been developed for controlling the quality of production processes (Sec. 23.5), for assessing the quality of produced goods (Sec. 23.6), for determining whether some function $F(x)$ is the unknown distribution function of some population (Sec. 23.7), and for situations in which the distribution of a population need not be known in order to perform a test (Sec. 23.8).

SOLUTIONS TO PROBLEM SET 23.4, page 1127

2. If the hypothesis $p = 0.5$ is true, X = *Number of heads in* 4040 *trials* is approximately normal with $\mu = 2020$, $\sigma^2 = 1010$ (Sec. 22.8). $P(X \leqq c) = \Phi([c - 2020]/\sqrt{1010}) = 0.95$, $c = 2072 > 2048$, do not reject the hypothesis.

4. Left-sided test, $\sigma^2/n = 9/20 = 0.45$. From Table A8 in Appendix 5 we obtain

$$P(\overline{X}) \leqq c)_{\mu=60.0} = \Phi\left(\frac{c - 60.0}{\sqrt{0.45}}\right) = 0.05.$$

Hence

$$c = 60.0 - 1.645\sqrt{0.45} = 58.9 > \bar{x}$$

and we reject the hypothesis.

6. We obtain

$$\eta(57.0) = P(\overline{X} \leqq c)_{\mu=57} = \Phi\left(\frac{58.9 - 57.0}{\sqrt{0.45}}\right) = \Phi(2.83) = 99\tfrac{3}{4}\%.$$

10. Hypothesis $\mu_0 = 35\,000$, alternative $\mu > 35\,000$. Using the given data and Table A9, we obtain

$$t = (37000 - 35000)/(5000/\sqrt{25}) = 2.00 > c = 1.71.$$

Hence we reject the hypothesis and assert that the manufacturer's claim is justified.

12. Hypothesis H_0: not better. Alternative H_1: better. Under H_0 the random variable

X = *Number of cases cured in* 400 *cases*

is approximately normal with mean $\mu = np = 300$ and variance $\sigma^2 = npq = 75$. From Table A8 and $\alpha = 5\%$ we get

$$(c - 300)/\sqrt{75} = 1.645, \qquad c = 300 + 1.645\sqrt{75} = 314.$$

Since the observed value 310 is not greater than c, we do not reject the hypothesis. This indicates that the results obtained so far do not establish the superiority.

14. We test the hypothesis $\sigma_0^2 = 25$ against the alternative that $\sigma^2 < 25$. As in Example 4, we now get

$$Y = (n - 1)S^2/\sigma_0^2 = 27S^2/25 = 1.08S^2.$$

From Table A10 with 27 degrees of freedom and the condition

$$P(Y < c) = \alpha = 5\% \qquad \text{we get} \qquad c = 16.2.$$

Since $y = 1.08s^2 = 1.08 \cdot 3.5^2 = 13.23 < c$ and the test is left-sided, we reject the hypothesis and assert that it will be less expensive to replace all the batteries simultaneously.

16. We test the hypothesis $\sigma_1^2 = \sigma_2^2$ against the alternative $\sigma_1^2 > \sigma_2^2$. We proceed as in Example 6. By computation,

$$v_0 = s_1^2/s_2^2 = 350/61.9 = 5.65.$$

For $\alpha = 5\%$ and (5, 6) degrees of freedom. Table A11 gives 4.39. Since 5.65 is greater, we reject the hypothesis and assert that the variance of the first population is greater than that of the second.

18. In this two-sided test we use (11), obtaining

$$t_0 = \sqrt{\frac{12 \cdot 18 \cdot 28}{30}} \cdot \frac{10 - 14}{\sqrt{11 \cdot 9 + 17 \cdot 9}} = -3.58.$$

From the t-table with $n_1 + n_2 - 2 = 28$ degrees of freedom we obtain $c_2 = 2.05$ corresponding to $97\frac{1}{2}\%$ and -2.05 for $2\frac{1}{2}\%$, by symmetry. Since $-3.58 < -2.05$, we reject the hypothesis and assert that the population means are different.

SECTION 23.5. Quality Control, page 1128

Purpose. Quality control is a testing procedure performed every hour (or every half hour, etc.) in an ongoing process of production in order to see whether the process is running properly ("is under control," is producing items satisfying the specifications) or not ("is out of control"), in which case the process is being halted in order to search for the trouble and remove it. These tests may concern the mean, variance, range, etc.

Main Content

Control chart for the mean

Control chart for the variance

Comment on Content

Control charts have also been developed for the range, the number of defectives, the number of defects per unit, for attributes, etc. (see the problem set).

SOLUTIONS TO PROBLEM SET 23.5, page 1132

2. $1 \pm 3 \cdot 0.02/\sqrt{4} = 1 \pm 0.03$

4. Decrease by a factor $\sqrt{2} = 1.41$. By a factor $2.58/1.96 = 1.32$. Hence the two operations have almost the same effect.

6. LCL = 3.5, UCL = 6.5

8. The sample range tends to increase with increasing n, whereas σ remains unchanged.

10. The random variable Z = *Number of defectives in a sample of size n* has the variance npq. Hence $\bar{X} = Z/n$ has the variance $\sigma^2 = npq/n^2 = 0.04 \cdot 0.96/100 = 0.000\,384$. This gives

$$\text{UCL} = 0.04 + 3\sigma = 0.0988.$$

From the given values we see that the process is not in control.

12. Choose 4 times the original sample size.

14. $\text{LCL} = n\mu_0 - 2.58\sigma\sqrt{n}$, $\text{UCL} = n\mu_0 + 2.58\sigma\sqrt{n}$, as follows from Theorem 1 in Sec. 23.3.

SECTION 23.6. Acceptance Sampling, page 1133

Purpose. This is a test for the quality of a produced lot designed to meet the interests of both the producer and the consumer of the lot, as expressed in the terms listed below.

Main Content, Important Concepts

Sampling plan, acceptance number, fraction defective

Operating characteristic curve (OC curve)

Acceptable quality level (AQL)

Rejectable quality level (RQL)

Rectification

Average outgoing quality limit (AOQL)

Comments on Content

Basically, acceptance sampling first leads to the hypergeometric distribution, which, however, can be approximated by the simpler Poisson distribution and simple formulas resulting from it, or in other cases by the binomial distribution, which can in turn be approximated by the normal distribution. Typical cases are included in the problem set.

SOLUTIONS TO PROBLEM SET 23.6, page 1136

2. We expect a decrease of values because of the exponential function in (3), which involves n. The probabilities are 0.9098 (down from 0.9825), 0.7358, 0.0404.

4. $P(A; \theta) = e^{-20\theta}(1 + 20\theta)$ from (3). From Fig. 504 we find α and β. For $\theta = 1.5\%$ we obtain $P(A; 0.015) = 96.3\%$, hence $\alpha = 3.7\%$. Also $\beta = P(A; 0.075) = 55.8\%$, which is very poor.

6. $[\theta e^{-30\theta}(1 + 30\theta)]' = 0$ gives $\theta = 0.054$ and the value 0.028.

8. The approximation is $\theta^0(1 - \theta)^2$ and is fairly accurate, as the following values show:

θ	Exact (2D)	Approximate
0.0	1.00	1.00
0.2	0.63	0.64
0.4	0.35	0.36
0.6	0.15	0.16
0.8	0.03	0.04
1.0	0.00	0.00

10. From the definition of the hypergeometric distribution we now obtain

$$P(A; \theta) = \binom{20\theta}{0}\binom{20 - 20\theta}{3}\bigg/\binom{20}{3} = \frac{(20 - 20\theta)(19 - 20\theta)(18 - 20\theta)}{6840}.$$

This gives $P(A; 0.1) = 0.72$ (instead of 0.81 in Example 1) and $P(A; 0.2) = 0.49$ (instead of 0.63), a decrease in both cases, as had to be expected.

12. $P(A; \theta) = e^{-20\theta}(1 + 20\theta)$. $[\theta P(A; \theta]' = 0$ gives $\theta = \theta_0 = 0.0809$, $\theta_0 P(A; \theta_0) = 0.0420$.

14. For $\theta = 0.05$ we should get $P(A; \theta) = 0.98$. (Figure 504 illustrates this, for different values.) Since $n = 100$, we get $np = 5$ and the variance $npq = 5 \cdot 0.95 = 4.75$. Using the normal approximation of the binomial distribution, we thus obtain, with c to be determined,

$$\sum_{x=0}^{c} \binom{100}{x} 0.05^x 0.95^{100-x} \approx \Phi\left(\frac{c - 5 + 0.5}{\sqrt{4.75}}\right) - \Phi\left(\frac{0 - 5 - 0.5}{\sqrt{4.75}}\right) = 0.98,$$

$$\Phi\left(\frac{c - 5 + 0.5}{\sqrt{4.75}}\right) = \Phi\left(\frac{-5.5}{\sqrt{4.75}}\right) + 0.98 = 0.9859.$$

From this and Table A8 we get (by interpolation)

$$c = 4.5 + 2.214\sqrt{4.75} = 9.325.$$

The *answer* is that we should choose 9 or 10 as c.

SECTION 23.7. Goodness of Fit. χ^2-Test, page 1137

Purpose. The χ^2-test is a test for a whole unknown distribution function, as opposed to the previous tests for unknown parameters in known types of distributions.

Main Content

Chi-square test

Test of normality

Comments on Content

The present method includes many practical problems, some of which are illustrated in the problem set.

Recall that the chi-square distribution also occurred in connection with confidence intervals and in our basic section on testing (Sec. 23.4).

SOLUTIONS TO PROBLEM SET 23.7, page 1140

2. $\chi_0^2 = 0.4 < c = 3.84$. Assert that the coin is fair.

4. $\chi_0^2 = 94.19 > 11.07$, reject. As usual, it is interesting to see the contributions of the various terms to χ_0^2. In the present case these vary considerably, between 1.6 and 52:

$$1.628 + 26.582 + 7.426 + 52.250 + 3.945 + 2.359.$$

6. $\chi_0^2 = \frac{1}{8}[(13 - 8)^2 + (3 - 8)^2 + (8 - 8)^2] = 6.25 > 3.84$ because $p = (13 + 3 + 8)/600 = 4\%$ was estimated, so that we have $K - 1 - 1 = 1$ degree of freedom. The difference between the numbers of defectives is significant.

8. The maximum likelihood estimates for the two parameters are $\bar{x} = 59.87$, $\hat{s} = 1.504$. $K - 1 - 2 = 2$ degrees of freedom. From Table 23.10 we get the critical value $9.21 > \chi_0^2 = 6.10$. Accept the hypothesis that the population from which the sample was taken is normally distributed. χ_0^2 is obtained as follows.

x	$\dfrac{x-\bar{x}}{s}$	$\Phi\left(\dfrac{x-\bar{x}}{s}\right)$	Expected	Observed	Terms in (1)
58.5	−0.91	0.1812	14.31	14	0.01
59.5	−0.25	0.4028	17.51	17	0.01
60.5	0.42	0.6623	20.50	27	2.06
61.5	1.08	0.8608	15.68	8	3.76
			11.00	13	0.36
					$\chi_0^2 = 6.20$

Slightly different results due to rounding are possible.

10. Let $50 + b$ be that number. Then $2b^2/50 > c$, $b > 5\sqrt{c}$, $50 + b = 60, 63, 64$.

12. $K = 2$ classes (dull, sharp). Expected values 10 dull, 390 sharp; 1 degree of freedom; hence

$$\chi_0^2 = \tfrac{49}{10} + \tfrac{49}{390} = 5.03 > 3.84.$$

Reject the claim. Two things are interesting here. First, 16 dull blades (an excess of 60% over the expected value!) would not have been sufficient to reject the claim at the 5% level. Second, 49/10 contributes much more to χ_0^2 than 49/390 does; in other applications the situation will often be qualitatively similar.

14. TEAM PROJECT. $n = 3 \cdot 77 = 231$.

(*a*) $a_j = 231/20 = 11.55$, $K = 20$, $\chi_0^2 = 24.32 < c = 30.14$ ($\alpha = 5\%$, 19 degrees of freedom). Accept the hypothesis.

(*b*) $\chi_0^2 = 13.10 > c = 3.84$ ($\alpha = 5\%$, 1 degree of freedom). Reject the hypothesis.

(*c*) $\chi_0^2 = 10.62 > c = 3.84$ ($\alpha = 5\%$, 1 degree of freedom). Reject the hypothesis.

SECTION 23.8. Nonparametric Tests, page 1142

Purpose. To introduce the student to the ideas of nonparametric tests in terms of two typical examples selected from a wide variety of tests in that field.

Main Content

Median, a test for it

Trend, a test for it

Comment on Content

Both tasks have not yet been considered in the previous sections. Another approach to trend follows in the next section.

SOLUTIONS TO PROBLEM SET 23.8, page 1143

2. $\frac{1}{2}^6 + 6 \cdot (\frac{1}{2})^6 + 15 \cdot (\frac{1}{2})^6 = 34\%$ is the probability of at most 2 negative values if $\tilde{\mu} = 0$, which we do not reject.

4. We drop 0 from the sample. Let X = *Number of positive values.* Under the hypothesis we get the probability

$$P(X = 9) = \binom{9}{9}\left(\frac{1}{2}\right)^9 = 0.2\%.$$

Accordingly, we reject the hypothesis that there is no difference between A and B and assert that the observed difference is significant.

6. Under the hypothesis the probability of obtaining at most 3 negative differences $(80 - 85, 90 - 95, 60 - 75)$ is

$$\left(\frac{1}{2}\right)^{15}\left[1 + \binom{15}{1} + \binom{15}{2} + \binom{15}{3}\right] = 1.76\%.$$

We reject the hypothesis and assert that B is better.

8. Let X = *Number of positive values among* 8 *values*. If the hypothesis is true, a positive value is as probable as a negative value and thus has probability 1/2. Hence, under the hypothesis the probability of getting at most 1 positive value is

$$P = \tfrac{1}{2}^{8} + 8 \cdot \tfrac{1}{2}^{8} = 3.5\%.$$

Hence we reject the hypothesis and assert that the setting is too low.

10. $n = 5$ values, with 2 transpositions, namely,

111.1 before 110.9 and 111.0,

so that from Table A12 we obtain

$$P(T \leqq 2) = 0.117$$

and we do not reject the hypothesis.

12. $n = 8$ values, with 4 transpositions, namely,

33.4 before 31.6
35.3 before 31.6, 35.0
37.6 before 36.5.

Table A12 gives

$$P(T \leqq 4) = 0.007.$$

Reject the hypothesis that the amount of fertilizer has no effect and assert that the yield increases with increasing amounts of fertilizer.

14. We order by increasing x. Then we have 10 transpositions:

$418 > 301, 352, 395, 375, 388$
$395 > 375, 388$
$465 > 455$
$521 > 455, 490$

Hypothesis *no trend*, alternative *positive trend*, $P(T \leqq 10) = 1.4\%$ by Table A12 in Appendix 5. Reject the hypothesis.

SECTION 23.9. Regression Analysis. Fitting Straight Lines, page 1145

Purpose. This section is a short introduction to regression analysis, restricted to linear regression and involving the famous least squares principle.

Main Content

Distinction between correlation and regression

Gauss's least squares method

Sample regression line, sample regression coefficient

Population regression coefficient, a confidence interval for it

SOLUTIONS TO PROBLEM SET 23.9, page 1150

2. $y = 2 - 0.55x$

4. $y = 2.99x$, $k = 1/2.99$

6. $y = -120.5 + 9.15x$, $y(35) = 200$. The negative constant -120.5 simply indicates that our linear interpolation by the least squares principle is meaningful only over a relatively short interval where we can approximate the actual function $y(x)$ by a linear function.

8. $\bar{x} = 2.5$, $s_x^2 = 5/3$, $\bar{y} = 7.475$, $s_y^2 = 14.9225$, $3s_{xy} = 14.95$, $k_1 = 2.99$, $q_0 = 0.067$, $c = 4.30$ (2 degrees of freedom) from (13) and the t-table, $K = 0.35197$, so that the *answer* is

$$\text{CONF}_{0.95}\{2.63 \leqq \kappa_1 \leqq 3.34\}.$$

10. Multiplying out the square, we get three terms, hence three sums,

$$\begin{aligned}\sum (x_j - \bar{x})^2 &= \sum x_j^2 - 2\bar{x}\sum x_j + n\bar{x}^2 \\ &= \sum x_j^2 - \frac{2}{n}\sum x_i \sum x_j + n\left(\frac{1}{n}\sum x_j\right)^2\end{aligned}$$

and the last of these three terms cancels half of the second term, giving the result.

SECTION 23.10. Correlation Analysis, page 1150

Purpose. Correlation analysis deals with the interrelation of X and Y in a two-dimensional random variable (X, Y). This section is an introduction without proofs.

Main Content, Important Concepts

Sample covariance s_{xy}

Sample correlation coefficient r

Population covariance σ_{XY}

Population correlation coefficient ρ

Independence of X and Y implies $\rho = 0$ ("uncorrelatedness").

Two-dimensional normal distribution

If (X, Y) is normal, $\rho = 0$ implies independence of X and Y.

Test for $\rho = 0$

SOLUTIONS TO CHAPTER 23 REVIEW, page 1153

26. $\bar{x} = 20.325$, $s^2 = 4.551$, $s = 2.133$

28. $\hat{\mu} = 20.325$, $\hat{\sigma}^2 = (7/8)s^2 = 3.982$

30. $k = 1.96 \cdot 5/\sqrt{500}$ (see Table 23.1 in Sec. 23.3). $\text{CONF}_{0.95}\{21.56 \leqq \mu \leqq 22.44\}$

32. $k = 2.576 \cdot 3.2/\sqrt{8} = 3.0$ (Table 23.1 in Sec. 23.3). $\text{CONF}_{0.99}\{28.4 \leqq \mu \leqq 34.4\}$

34. $k = 2.06 \cdot 7/\sqrt{2} = 2.9$ from the t-table in Appendix 5 with 24 degrees of freedom. $\text{CONF}_{0.95}\{113.1 \leqq \mu \leqq 118.9\}$

36. $n - 1 = 3$ degrees of freedom, $F(c_1) = 0.025$, $c_1 = 0.22$, $F(c_2) = 0.975$, $c_2 = 9.35$ from Table A10 in Appendix 5; hence $k_1 = 0.05/0.22 = 0.227$, $k_2 = 0.05/9.35 = 0.005$

by Table 23.3 in Sec. 23.3. The answer is

$$\text{CONF}_{0.95}\{0.005 \leqq \sigma^2 \leqq 0.227\}.$$

38. The test is two-sided. We have $\sigma^2/n = 0.025$, as before. Table A8 gives

$$P(\overline{X} < c)_{15.0} = \Phi\left(\frac{c - 15.0}{\sqrt{0.025}}\right) = 0.975, \qquad c = 15.31$$

and $15.0 - 0.31 = 14.69$ as the left endpoint of the acceptance region. Now $\bar{x} = 14.5 < 14.7$, and we reject the hypothesis.

40. We proceed as in Example 3 in Sec. 23.4. The test is right-sided. From Table A9 with $n - 1 = 19$ degrees of freedom and

$$P(T > c)_{\mu_0} = 0.01, \qquad \text{thus} \qquad P(T \leqq c)_{\mu_0} = 0.99$$

we get $c = 2.54$. From the sample we compute

$$t = \frac{29.8 - 28.0}{\sqrt{1.2}/\sqrt{20}} = 7.35 > c$$

and reject the hypothesis.

42. $\bar{x} = 376.3$, $\bar{y} = 335.3$, $s_1^{\,2} = 1009.3$, $s_2^{\,2} = 869.3$, $t_0 = 1.64 < c = 2.92$ ($\alpha = 5\%$, 2 degrees of freedom); do not reject the hypothesis.

44. Because the sample size n is finite.

46. $\alpha = 1 - (1 - \theta)^6 = 5.85\%$, when $\theta = 0.01$. For $\theta = 15\%$ we obtain $\beta = (1 - \theta)^6 = 37.7\%$. If n increases, so does α, whereas β decreases.

48. We drop the two rods of exact length. Then we have a sample of 18. Under the hypothesis that no adjustment is needed, longer rods and shorter rods have the same probability $\frac{1}{2}$. Hence the probability of getting three or fewer shorter rods is

$$(\tfrac{1}{2})^{18}(1 + 18 + 153 + 816) = 0.0038$$

and we reject the hypothesis and accept the alternative.

50. $y = 3.4 - 1.85x$